Unit-I

Partial Differential Equation

Definition:

A partial differential Equation is an equation involving a function of two or more variables and some of its partial derivatives

Examples:

ordinary Differential Eqn 1) General equation is $y = x^2 + 3x + 2 \Rightarrow y = f(x)$

$\downarrow$ Here x is an independent variable

$\frac{dy}{dx}$ (1) y is an dependent variable

$\frac{d^2y}{dx^2}$ $\Rightarrow$ One Dependent variable and one Independent variable. Then only we can differentiatio

2) $z = 3x^2 - 9y$ $\frac{\partial z}{\partial x}$

Which one is an Partial differentiation and More than one Independent variable

Formation of Partial Differential Equation:

It can split up into two types They are

1) Eliminating Arbitary Constant (a, b)
2) Eliminating Arbitary function (u, v)

Notations:

$z \rightarrow$ Dependent variables

$x, y \rightarrow$ Independent variables

$$\frac{\partial z}{\partial x} = p; \quad \frac{\partial z}{\partial y} = q; \quad \frac{\partial^2 z}{\partial x^2} = r; \quad \frac{\partial^2 z}{\partial x \partial y} = s; \quad \frac{\partial^2 z}{\partial y^2} = t$$

I **Problems:**

1) Form the PDE by eliminating the arbitary Constant a & b From $z = (x^2 + a)(y^2 + b)$

Given $z = (x^2+a)(y^2+b) \rightarrow (1)$

Differentiate z partially with respect to x, y we get

$$\Rightarrow \frac{\partial z}{\partial x} = 2x(y^2+b)$$

$$\Rightarrow P = 2x(y^2+b) \rightarrow ② \Rightarrow y^2+b = P/2x$$

$$\Rightarrow \frac{\partial z}{\partial y} = 2y(x^2+a).$$

$$\Rightarrow q = 2y(x^2+a) \rightarrow ③$$

$$\Rightarrow x^2+a = q/2y$$

Then Substitute these in ① we get

$$z = P/2y \; q/2x = \frac{pq}{4xy}$$

$$\Rightarrow \boxed{4xy\,z = pq}$$

Hence the Proved.

② Find the PDE by eliminating the arbitrary constant a & b from $z = ax + by$

Soln:

$$z = ax + by \rightarrow ①$$

Diff. Partially with respect to x we get

$$\frac{\partial z}{\partial x} = a \Rightarrow P = a \rightarrow ②$$

Diff. Partially with respect to y weget

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \rightarrow ③$$

Then Substitute these eqn in ① we get

$$\boxed{z = Px + qy}$$

Hence the proved

③ Find the PDE by eliminating the arbitrary constant from $(x-a)^2 + (y-b)^2 + z^2 = 1$

Soln:

Given $(x-a)^2 + (y-b)^2 + z^2 = 1 \rightarrow ①$

Diff. partially with respect to x

$$2(x-a)' + 2z\frac{\partial z}{\partial x} = 0$$

$$\cancel{2}(x-a) = -\cancel{2}z\frac{\partial z}{\partial x}$$

$$(x-a) = -zp \rightarrow (2)$$

Diff. partially with respect to 'y'

$$0 + 2(y-b)' + 2z\frac{\partial z}{\partial y} = 0$$

$$\cancel{2}(y-b) = -\cancel{2}\cancel{z}\frac{\partial z}{\partial y}$$

$$(y-b) = -zq \rightarrow (3)$$

Then substitute these in eqn (1) we get

$$(x-a)^2 + (y-b)^2 + z^2 = 1$$

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

$$z^2p^2 + z^2q^2 + z^2 = 1$$

$$\boxed{z^2(p^2+q^2+1) = 1}$$

Hence the proved.

(4) **Find the PDE of all planes cutting equal intercepts from the x & y axis?**

Let a, c be the intercepts on x & y axis respectiv

Hence the equation of the plane is $\frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \rightarrow (1)$

Diff. (1) p.w. to x and y we get

$$1/a + 1/c\,\frac{\partial z}{\partial x} = 0 \Rightarrow 1/a + p\,1/c = 0 \rightarrow (2)$$

$$1/a + 1/c\,\frac{\partial z}{\partial y} = 0 \Rightarrow 1/a + q\,1/c = 0 \rightarrow (3)$$

$$(2) - (3) \Rightarrow 1/c\,(p-q) = 0$$

$$\Rightarrow (p-q) = 0$$

II By elimination of arbitary function:

The elimination of one arbitary function from a given relation gives a PDE of first order while elimination of two arbitary function from a given relation gives second or higher order partial differential equations

Problem:

1) From the PDF by eliminating the arbitary function $z = f(x^2+y^2)$ 1^{st} order

Given $z = f(x^2+y^2) \rightarrow ①$

Diff. Partially with respect to x and y we g

$$p = \frac{\partial z}{\partial x} = f'(x^2+y^2)2x \rightarrow ②$$

$$q = \frac{\partial z}{\partial y} = f'(x^2+y^2)2y \rightarrow ③$$

$$② \div ③ \Rightarrow \frac{p}{q} = \frac{x}{y}$$

$$(py - qx) = 0$$

Hence the proved

2) From the PDE by eliminating the arbitary function $z = f(x+ct) + \phi(x-ct) \rightarrow 2^{nd}$ order

Given $z = f(x+ct) + \phi(x-ct) \rightarrow ①$

Diff. Partially with respect to "x" and "t" we get

$$z_x = \frac{\partial z}{\partial x} = p = f'(x+ct).(1) + \phi'(x-ct)(1)$$

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = r = f''(x+ct).(1) + \phi''(x-ct)(1) \rightarrow ②$$

$$z_{xx} = f''(x+ct) + \phi''(x-ct) \rightarrow ②$$

$$p_t = \frac{\partial z}{\partial t} = q = f'(x+ct)(c) + \phi'(x-ct)(-c)$$

$$t_{tt} = \frac{\partial^2 z}{\partial t^2} = f''(x+ct)c^2 + \phi'(x-ct)c^2$$

$$z_{tt} = c^2\left(f''(x+ct) + \phi'(x-ct)\right) \rightarrow 3$$

Sub 2 in 3 we get

$$z_{tt} = c^2(z_{xx})$$

$$\boxed{t = c^2 r}$$

Hence The proved

3) H.W.: From The PDE by eliminating The arbitary function $z = f(x+at) + g(x-at)$ 2nd order eqn

Ans: $z_{tt} = a^2 z_{xx}$

4) From The PDE by eliminating The arbitary function ϕ & ψ from 2nd order $z = \phi(x+iy) + \psi(x-iy)$

Given $z = \phi(x+iy) + \psi(x-iy) \rightarrow 1$

W.K.T $i = \sqrt{-1}$

Diff. partially with respect to x and y we get

Here $i = \sqrt{-1}$ and $i^2 = -1$

$$\frac{\partial z}{\partial x} = \phi'(x+iy)(1) + \psi'(x-iy)(1)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \phi''(x+iy)(1) + \psi''(x-iy)(1) \rightarrow 2$$

$$\frac{\partial z}{\partial y} = \phi'(x+iy)(i) + \psi'(x-iy)(-i)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \phi''(x+iy)(i^2) + \psi'(x-iy)(i^2)$$

$$= i^2\left[\phi''(x+iy) + \psi'(x-iy)\right]$$

$$= -1\left[\phi''(x+iy) + \psi'(x-iy)\right] \rightarrow 3$$

Sub 2 in 3 weget

$$t = -r$$

$$\boxed{t + r = 0}$$ Hence The proved

5 From the PDE by eliminating f from

$$z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$$

Given $z = x^2 + 2f\left(\frac{1}{y} + \log x\right) \rightarrow (1)$

Diff. Partially with respect to x

$$p = \frac{\partial z}{\partial x} = 2x + 2f'\left(\frac{1}{y} + \log x\right)\frac{1}{x} \rightarrow (2)$$

Diff. Partially with respect to y

$$q = \frac{\partial z}{\partial y} = 2f'\left(\frac{1}{y} + \log x\right)\left(-1y^{-2}\right)$$

$$= 2f'\left(\frac{1}{y} + \log x\right)\left(-\frac{1}{y^2}\right) \rightarrow (3)$$

From (2) $\times x \Rightarrow px = 2x^2 + 2f'\left(\frac{1}{y} + \log x\right) \rightarrow (4)$

From (3) $\times y^2 \Rightarrow qy^2 = 2f'\left(\frac{1}{y} + \log x\right) - \cancel{y^2}/\cancel{y^2} \rightarrow (5)$

(4) + (5) $\Rightarrow px + qy^2 = 2x^2$

Hence the proved

III Formation of Partial Differential equation by elimination of arbitary function ϕ from $\phi(u, v) = 0$ where 'u' & 'v' are function of x, y and z

Let $\phi(u, v) = 0 \rightarrow (1)$ be the function of u and v where u and v are the function of x, y & z

Diff (1) partially with respect to x, y we g

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0 \rightarrow (2)$$

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = 0 \rightarrow (3)$$

Here we want to eliminate ϕ.

To eliminate ϕ it is eliminate to $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$

From ② and ③

Hence we get $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \rightarrow (4)$

where $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are to be determined from u & v.

Problems :

1) **From the PDE by eliminating the arbitary functions from the relation $\phi(x^2+y^2+z^2, lx+my+nz) = 0$**

Soln:

Given $\phi(x^2+y^2+z^2, lx+my+nz) = 0$

Let $u = x^2+y^2+z^2$ $\quad v = lx+my+nz$

$\therefore$ ① is the form of $\phi(u,v) = 0$

$\frac{\partial u}{\partial x} = 2x+2zp$; $\frac{\partial v}{\partial x} = l+np$

$\frac{\partial u}{\partial y} = 2y+2zq$; $\frac{\partial v}{\partial y} = m+nq$

$\therefore \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 2x+2zp & l+np \\ 2y+2zq & m+nq \end{vmatrix}$

$\Rightarrow (2x+2zp)(m+nq) - (2y+2zq)(l+np) = 0$

$\Rightarrow 2xm + 2xnq + 2zpm + 2znpq - 2yl - 2ynp - 2zql - 2zqnp = 0$

$\Rightarrow (mz-ny)p + (nx-lz)q = yl - mx$ which is the required eqn.

Hence the proved

2) From the PDE by eliminating f from

$f(x^2+y^2+z^2, x+y+z) = 0$

Given $f(x^2+y^2+z^2, x+y+z) = 0 \rightarrow (1)$

Let $u = x^2+y^2+z^2$, $v = x+y+z$

$\therefore$ (1) is of the form $f(u, v=0) \rightarrow (2)$

$$\frac{\partial u}{\partial x} = 2x + 2zp \; ; \; \frac{\partial v}{\partial x} = 1+p$$

$$\frac{\partial u}{\partial y} = 2y + 2zq \; ; \; \frac{\partial v}{\partial y} = 1+q$$

$$\Rightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2x+2zp & 1+p \\ 2y+2zq & 1+q \end{vmatrix} = 0$$

$$\Rightarrow (2x+2zp)(1+q) - (2y+2zq)(1+p) = 0$$

$$\Rightarrow 2(x+zp)(1+q) - 2(y+zq)(1+p) = 0$$

$$\Rightarrow (x+zp)(1+q) - (y+zq)(1+p) = 0$$

$$\Rightarrow x + xq + zp + zpq - y - yp - zq - zpq = 0$$

$$\Rightarrow (z-y)p + (x-z)q = y - x$$ This is

The required solution.

1.4 **Definition:**

The eqn $Pp + Qq = R$ is called Lagrange's linear equation.

Here $P = \frac{\partial(u,v)}{\partial(y,z)}$ $Q = \frac{\partial(u,v)}{\partial(z,x)}$ $R = \frac{\partial(u,v)}{\partial(x,y)}$

Solution of PDE by Direct integration Method.

1) **Solve** $\frac{\partial^2 z}{\partial x^2} = 0$ (formula sum)

Given $\frac{\partial^2 z}{\partial x^2} = 0$

Integrating w.r.t x taking y as constant

$\frac{\partial z}{\partial x} = c_1 \Rightarrow \frac{\partial z}{\partial x} = F(y)$

Integrating w.r.t x again taking y as a constant

formula sum

$z = f(y) \cdot x + c_2$

$z = f(y) \cdot x + g(y)$

Here $F(y)$ and $g(y)$ are some arbitary constant of y

Hence the proved

2) **Solve** $\frac{\partial^2 z}{\partial y^2} = \sin(2x + 3y)$

Given $\frac{\partial^2 z}{\partial y^2} = \sin(2x + 3y)$

Integrating w.r.t y taking x as constant

$\frac{\partial z}{\partial y} = -\cos(2x+3y) \cdot \frac{1}{3} + c_1$

$\frac{\partial z}{\partial y} = -\cos(2x+3y) \cdot \frac{1}{3} + f(x)$

Sing w.r.t y again taking x as a constant

$z = -\sin(2x+3y) \cdot \frac{1}{9} + f(x) \cdot y + c_2$

$z = -\sin(2x+3y) \cdot \frac{1}{9} + f(x) \cdot y + g(x)$

Here $f(x)$ and $g(x)$ are some arbitary function of x.

Hence the proved

(3) Solve $\frac{\partial^2 z}{\partial x^2} = xy$

Given $\frac{\partial^2 z}{\partial x^2} = xy$

on Sing w.r.t x taking y as a constant

$$\frac{\partial z}{\partial x} = \frac{x^2}{2} y + c_1$$

$$\frac{\partial z}{\partial x} = \frac{x^2}{2} y + f(y)$$

on Sing w.r.t x taking y as a constant

$$z = \frac{x^3}{6} y + f(y) \cdot x + c_2$$

$$z = \frac{x^3}{6} y + f(y) \cdot x + g(y)$$

Here $f(y)$ and $g(y)$ are some arbitary function of y.

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(3) H.W $\frac{\partial^2 z}{\partial x^2} = \sin x$

Definition:

A Solution in which The number of arbitary constant is equal to The number of Independent Variables is called complete integral or Complete solution.

Definition:

Let $f(x, y, z, p, q) = 0$ be a PDE of complete integral is $\phi(x, y, z, a, b) = 0 \rightarrow (1)$

Diff. P. w.t a, b Then equal to zero That is

$$\frac{\partial \phi}{\partial a} = 0 \rightarrow (2) \qquad \frac{\partial \phi}{\partial b} = 0 \rightarrow (3)$$

The elimination of a & b from the above three equation is called Singular Integral

IV Method to solve the first order PDE

Type I: $f(p,q)=0$ Single Integral Method:

Instructions:

1) Equation has only p and q
2) x, y, z are missing
3) Substitute $p=a$ and $q=b$
4) write b in terms of a
5) Substitute the value of b in $z = ax + by + c$
6) No singular integral for this type.
7) Find General Solution

Problem i) Singular Integral:

1) $pq = 1$ Solve?

$\Rightarrow$ Type I :: $f(p,q)=0$ $\Rightarrow x, y, z$ are missing

$\Rightarrow$ Put $p=a$ and $q=b$,

we get $ab=1$

$\Rightarrow \boxed{b = 1/a}$

$\Rightarrow$ so, $z = ax + \left(\frac{1}{a}\right)y + c$ is the complete solution

$\Rightarrow$ Let us take $c = \phi(a)$ so we get $z = ax + \left(\frac{1}{a}\right)y + \phi(a) \rightarrow (1)$

Diff eqn (1) P.w.r.t 'a' we get

$$0 = x + a^{-2}y + \phi'(a) \rightarrow (2)$$

Then eliminating of 'a' between eqns (1) and (2) we will give the general solution

2) $p^2 + q^2 = 1$

$\Rightarrow$ Type I: $f(p,q)=0$ $\Rightarrow x, y, z$ are missing

$\Rightarrow$ Put $p=a$ and $q=b$

we get $a^2+b^2=1 \Rightarrow b^2 = 1-a^2 \Rightarrow \boxed{b = \sqrt{1-a^2}}$

$\Rightarrow$ so, $z = ax + \sqrt{1-a^2}\, y + c$ is the complete soln.

$\Rightarrow$ Let us take $c = \phi(a)$ so we get

$$z = ax + \sqrt{1-a^2}\, y + \phi(a) \rightarrow ①$$

Diff eqn ① P. w.r.t 'a' we get

$$0 = x + \sqrt{1-a^2}\, y + \phi'(a)$$

$$= x + (1-a^2)^{1/2} \cdot y + \phi'(a)$$

$$= x + \tfrac{1}{2}\left((1-a^2)^{1/2}\right)^{-1} \times -2a \cdot y + \phi(a)$$

$$= x - \frac{a \cdot y}{\sqrt{1-a^2}} + \phi'(a) \rightarrow ②$$

The eliminant of 'a' between eqn ① and ② we will get the general soln.

③ $\sqrt{p} + \sqrt{q} = 1$

$\Rightarrow$ Type I :: $f(p,q) = 0$ $\quad$ x, y, z are missing

$\Rightarrow$ Put $p = a^2$ and $q = b^2$

we get $\sqrt{a^2} + \sqrt{b^2} = 1 \Rightarrow (a^2)^{1/2} + (b^2)^{1/2} = 1$

so $a + b = 1$

$$\boxed{b = 1 - a}$$

$\Rightarrow$ so, $z = ax + (1-a)y + c$ is the Complete Solution

$\Rightarrow$ Let us take $c = \phi(a)$ so we get

$$z = ax + (1-a)y + \phi(a) \rightarrow ①$$

Diff eqn ① P. w.r.t 'a' we get

$$0 = x + (y - ay) + \phi'(a)$$

$$= x + (-y) + \phi'(a)$$

$$= x - y + \phi'(a) \rightarrow ②$$

The eliminant of 'a' between eqn ① and ② we will get the general solution

Type 2 : $Z = px + qy + f(p,q)$ (Clairauts form)

① $z = px + qy + pq$

Given $z = px + qy + pq \rightarrow ①$

① To find complete Integral First to solve singular

Let $Z = ax + by + c \rightarrow ②$ be the solution

Diff ② w.r.t "x"

$$\frac{\partial z}{\partial x} = a$$

$$\boxed{p = a}$$

Diff ② w.r.t "y"

$$\frac{\partial z}{\partial y} = b$$

$$\boxed{q = b}$$

Substituting for p and q in eqn ①

$$① \Rightarrow \quad z = px + qy + pq$$

$$z = ax + by + ab \rightarrow ③ \quad \{\text{complete integral}\}$$

Now To find singular Integral differentiate ③ w.r.t a

$$Z = ax + by + ab$$

$$\frac{\partial z}{da} = x + b$$

$$\Rightarrow x + b = 0$$

$$\boxed{b = -x}$$

Now diff ③ w.r.t b

$$z = ax + by + ab$$

$$\frac{\partial z}{\partial b} = y + a$$

$$\Rightarrow y + a = 0$$

$$\boxed{a = -y}$$

Substituting for a and b in ③ we get singular integral

$$③ \Rightarrow z = ax + by + ab$$

$$z = -yx + (-x)y + (-x)(-y)$$

$$z = -xy - \cancel{xy} + \cancel{xy}$$

$z = -xy$ is the singular Integral

① Solve $z = px + qy + p^2q^2$

This eqn is of the form of Type II

$\therefore z = px + qy + f(p,q)$

Then the Complete soln is $z = ax + by + a^2b^2$ → ①

diff ① p.w.r.t 'a'

$0 = x + 0 + 2ab^2$

$x + 2ab^2 = 0$ → ② $\boxed{x = -2ab^2}$

diff ① p.w.r.t 'b'

$0 = 0 + y + a^2(2b)$

$y + 2a^2b = 0$ → ③ $\boxed{y = -2a^2b}$

Now eqn ① as ⇒ $z = ax + by + a^2b^2$

$z = a(-2ab^2) + b(-2a^2b) + a^2b^2$

$= -2a^2b^2 - 2a^2b^2 + a^2b^2$

$\boxed{z = -3a^2b^2}$ ⇒ $z^3 = -27a^6b^6$ → ④

Already w.k.T $x = -2ab^2$ and $y = -2a^2b$

⇒ $xy = 4a^3b^3$

⇒ $x^2y^2 = 16a^6b^6$

⇒ $\frac{x^2y^2}{16} = a^6b^6$ → ⑤

Sub ⑤ in ④ we get

⇒ $z^3 = -27\frac{x^2y^2}{16}$

⇒ $\boxed{16z^3 = -27x^2y^2}$ → ⑥

eqn ⑥ This the Singular integral

Now $b = \phi(a)$

Now eqn ① be

$z = ax + \phi(a)y + a^2[\phi(a)]^2$ → ⑦

diff ④ p.w.r.t 'a'

$0 = x + \phi'(a)y + 2a[\phi(a)]^2 + a^2 2\phi(a)\phi'(a)$ → 8

From (7) and (8) Eliminating 'a' we get the general solution.

Type 3: $f(z,p,q)=0$; $f(x,p,q)=0$; $f(y,p,q)=0$

$f(z,p,q)=0 \Rightarrow$ Let $z=f(x+ay)$ be the solution

Put $x+ay=u$

$z=f(u)$

Substitute $p=\frac{dz}{du}$, $q=a\frac{dz}{du}$

Then integrating to get the solution

1) $z=p+q$

Let $u=x+ay \rightarrow (1)$

Put $p=\frac{dz}{du}$, $q=a\frac{dz}{du}$

$\therefore z=\frac{dz}{du}+a\frac{dz}{du}=\frac{dz}{du}(1+a)$

$\Rightarrow \frac{dz}{du}=\frac{z}{1+a}$

$(1+a)\frac{dz}{z}=du$

On $\int$ ing we get

$\Rightarrow (1+a)\int\frac{dz}{z}=\int du$

W.K.T $\int \frac{1}{x}dx=\log x$ and $\int dx=x$

$\Rightarrow (1+a)\log z=u+b$

$\Rightarrow (1+a)\log z=(x+ay)+b$ $(\because$ by (1)$)$

2) Solve $z^2=1+p^2+q^2$

Let $u=x+ay \rightarrow (1)$

Put $p=\frac{dz}{du}$, $q=a\frac{dz}{du}$

$\therefore z^2=1+\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2$

$$\Rightarrow z^2 - 1 = \left(\frac{dz}{du}\right)^2 (1+a^2)$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{z^2-1}{(1+a^2)}$$

$$\Rightarrow \frac{dz}{du} = \sqrt{\frac{z^2-1}{1+a^2}}$$

$$\Rightarrow \sqrt{1+a^2}\,\frac{dz}{\sqrt{z^2-1}} = du$$

on $\int$ing we get

$$\Rightarrow \sqrt{1+a^2}\int\frac{dz}{\sqrt{z^2-1}} = \int du$$

W.K.T $\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}(x)$

$$\Rightarrow \sqrt{1+a^2}\,\cosh^{-1}(z) = u + b$$

$$\Rightarrow \sqrt{1+a^2}\,\cosh^{-1}(z) = (x+ay)+b$$

Lagranges Linear Equation:

A linear partial differential equation of the first order know as lagranges linear Equation is of the form $Pp + Qq = R$ where P, Q, R are function of x, y, z

Step 1 :: General form $Pp + Qq = R$

Step 2 :: Auxilary equation $\frac{dx}{P} = \frac{dy}{P} = \frac{dz}{R}$

Step 3 :: This has two equation they are grouping method and Method of multiplier

(i) grouping Method

$$\Rightarrow \frac{dx}{p} = \frac{dy}{q}$$

$$\Rightarrow \frac{dy}{q} = \frac{dz}{r}$$

(ii) Method of multiplier

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{0}$$

Put denominator as

Step 4: General Solution $\phi(u,v)$

Problem:

1) Solve $px + qy = z$

$\Rightarrow$ Lagrange Linear P.D.E

$$Pp + Qq = R$$

$$P = x \qquad Q = y \qquad z = R \rightarrow (1)$$

$\Rightarrow$ Auxilary Equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dy}{R}.$$

by (1), consider $\frac{dx}{P} = \frac{dy}{Q}$

$$\frac{dx}{x} = \frac{dy}{y}.$$

on $\int$ing both sides we get

$$\Rightarrow \log x = \log y + \log c_1$$

$$\Rightarrow \log x - \log y = \log c_1$$

$\{\log a - \log b = \log(a/b)\}$ $\Rightarrow \cancel{\log}\left(\frac{x}{y}\right) = \cancel{\log} c_1$

$$\therefore c_1 = x/y$$

$$\therefore \boxed{u = x/y}$$

by (1) consider $\frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$

on $\int$ing we get both sides

$$\log y = \log z + \log c_2$$

$$\log y - \log z = \log c_2$$

$$\cancel{\log}\left(\frac{y}{z}\right) = \cancel{\log} c_2$$

$$c_2 = y/z$$

$$\boxed{v = y/z}$$

(2) $P\tan x + q\tan y = \tan z$

$\Rightarrow$ lagrange P.D.E

$$Pp + Qq = R$$

$$P = \tan x \quad Q = \tan y \quad R = \tan z \rightarrow (1)$$

$\Rightarrow$ Auxilary Equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

by (1) Consider $\frac{dx}{P} = \frac{dy}{Q}$

$$\Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Simplified $\cot x\, dx = \cot y\, dy$

on $\int$ing both sides we get

$$\int \cot x\, dx = \int \cot y\, dy$$

$$\log \sin x = \log \sin y + \log c_1$$

$$\log\left(\frac{\sin x}{\sin y}\right) = \log c_1$$

$$\frac{\sin x}{\sin y} = c_1$$

$$\boxed{u = \frac{\sin x}{\sin y}}$$

by (1) Consider $\frac{dy}{\tan y} = \frac{dz}{\tan z}$

$$\int \cot y\, dy = \int \cot z\, dz$$

$$\log \sin y = \log \sin z + \log c_2$$

$$\log\left(\frac{\sin y}{\sin z}\right) = \log c_2$$

$$\boxed{v = \frac{\sin y}{\sin z}}$$

$\therefore$ The General solution is $\phi(u,v) \Rightarrow \phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

Homogeneous And Non-Homogeneous P.D.E

$$Z = C.F + P.I$$

C.F = Complementary function

P.I = Particular Integral

To find C.F :

Let $(aD^2 + bDD' + cD'^2)z = f(x,y)$

Replace D by m and D' by 1

$\therefore am^2 + bm + c = 0$

Then solve and find roots m_1 and m_2

If $m_1 \neq m_2$, $C.F = f_1(y + m_1 x) + f_2(y + m_2 x) + \dots$

If $m_1 = m_2$, $C.F = f_1(y + m_1 x) + x f_2(y + m_2 x) + \dots$

$M \cdot m_1 = m_2, \; m_3 \neq m_4$

$Z = f_1(y+m_1x) + x f_2(y+m_1x) + f_3(y+m_3x) + f_4(y+m_4x)$

Type 1: Right Hand Side = 0 (Homogeneous)

1) Solve $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 0$

The eqn can be written as $(D^2 - 3DD' + 2D'^2)z = 0$

Replace D by m and D' by 1

Then the A.E is $m^2 - 3m + 2 = 0$

$(m-1)(m-2) = 0$

$\boxed{m_1 = 1}$ $\boxed{m_2 = 2}$

Here $m_1 \neq m_2$ $\therefore C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$

$z = f_1(y + x) + f_2(y + 2x)$

Here R.H.S = 0 so we have not to find P.I

2) Solve $(2D^2 + 5DD' + 2D'^2)z = 0$

Replace D by m and D' by 1

Then the A.E is $2m^2 + 5m + 2 = 0$

$$(m + 1/2)(m + 4/2) = 0$$

$\boxed{m_1 = -1/2}$ $\boxed{m_2 = -2}$

$2m^2$ 4 $\boxed{1/2 \;|\; 4/2}$ 5

Here $m_1 \neq m_2$

$\therefore$ C.F $= f_1(y + m_1 x) + f_2(y + m_2 x)$

or

C.F $= f_1(y - 1/2\,x) + f_2(y - 2x)$

Here P.I $= 0$

$$\therefore z = f_1(y - 1/2\,x) + f_2(y - 2x).$$

---

3) Solve $(D^3 - 3DD'^2 + 2D'^3)z = 0$

Replace D by m and D' by 1

Then the A.E is $m^3 - 3m + 2 = 0 \rightarrow (1)$

Here the power is 3

So put $\boxed{m_1 = 1}$ $\Rightarrow 1 - 3 + 2 = 0$

by (1) Hence

$$\begin{array}{c|cccc} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$\rightarrow$ (Coefficient of m)

$\Rightarrow m^2 + m - 2 = 0$

$\Rightarrow (m - 1)(m + 2) = 0$

-2 $\boxed{-1 \;|\; 2}$ 1

$\Rightarrow \boxed{m_2 = 1}$ $\boxed{m_3 = -2}$

$\Rightarrow m_1 = 1 \quad m_2 = 1 \quad m_3 = -2$

C.F $= f_1(y + x) + x f_2(y + x) + f_3(y - 2x)$

Here P.I = 0

$\therefore z = f_1(y+x) + x f_2(y+x) + f_3(y-2x)$

Type 2 RHS = e^{ax+by}

1) Solve $\dfrac{\partial^2 z}{\partial x^2} - 5\dfrac{\partial^2 z}{\partial x \partial y} + 6\dfrac{\partial^2 z}{\partial y^2} = e^{x+y}$

$\frac{\partial z}{\partial x} = D$

$\frac{\partial^2 z}{\partial x^2} = D^2$

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y}$

Soln:

The eqn canbe written as,

$(D^2 z - 5DD'z + 6D'^2 z) = e^{x+y}$

$(D^2 - 5DD' + 6D'^2)z = e^{x+y}$

Replace D by m and D' by 1

Then the A.E is $m^2 - 5m + 6 = 0$

$(m-2)(m-3) = 0$

$\boxed{m_1 = 2}$ $\boxed{m_2 = 3}$

6

-2 -3

-5

Here $m_1 \neq m_2$

$\therefore$ C.F = $f_1(y+2x) + f_2(y+3x) \rightarrow (1)$

Now we have to find P.I

$P.I = \dfrac{1}{D^2 - 5DD' + 6D'^2} \cdot e^{x+y}$

$\left.\begin{matrix} x \to 1 \\ y \to 1 \end{matrix}\right\}$ coeffi

$D = 1$

$D' = 1$

$= \dfrac{1}{1-5+6} e^{x+y}$

$P.I = \dfrac{1}{2} e^{x+y} \rightarrow (2)$

$\therefore z = C.F + P.I$

$z = \left[f_1(y+2x) + f_2(y+3x)\right] + \left[\frac{1}{2} e^{x+y}\right]$

Hence the proved

(2) Solve $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

Replace D by m and D' by 1

Then the A.E is $m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$\boxed{m_1 = 2}\ \boxed{m_2 = 2}$

Here $m_1 = m_2$

so $C.F = f_1(y+2x) + x f_2(y+2x) \rightarrow (1)$

Now we have to find P.I

$$P.I = \frac{1}{D^2 - 4DD' + 4D'^2} \cdot e^{2x+y}$$

$D \Rightarrow x = 2$
$D' \Rightarrow y = 1$

$$= \frac{1}{4 - 4(2) + 4} \cdot e^{2x+y}$$

$$= \frac{1}{0} \cdot e^{2x+y}$$ (ordinary rule fail)

Then we Diff. 'D' $\left(\because D \rightarrow x,\ D' \Rightarrow y\right)$

$$= \frac{x}{2D - 4D'} \cdot e^{2x+y}$$

$$= \frac{x}{4-4} e^{2x+y}$$

$$= \frac{x}{0} \cdot e^{2x+y}$$ (ordinary rule fail)

Then onceagain diff w.r.t D' $\left(D \rightarrow x,\ D' \rightarrow y\right)$

$$P.I = \frac{x^2}{2} e^{2x+y} \rightarrow (2)$$

$\therefore z = C.F + P.I$

Unit - 2

Transforms and PDE

Fourier Series

* $(0, 2\pi)$
* $(0, 2l)$
* odd or even function
  $(-\pi, \pi)$ $(-l, l)$
* Half range cosine series $(0, l)$ $(0, \pi)$
* Half range Sine series $(0, l)$ $(0, \pi)$
* Complex form
* Harmonic Analysis

2m *

Dirichlet condition (or) The Sufficient condition for Fourier series:

Any function $f(x)$ can be expressed as fourier series in $(c, c+2\pi)$ (or) $(c, c+2l)$

* $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Same $\{\cos 30^\circ, \cos(2\pi + 30^\circ)\}$ $\{f(x) = x, f(4) = 4\}$ Max

i) $f(x)$ must be Periodic, Single value function, and finite in $(c, c+2\pi)$

ii) $f(x)$ has finite number of Maxima & Minima in $(c, c+2\pi)$

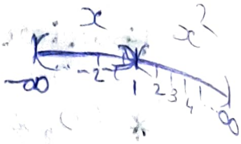
iii) $f(x)$ has finite number of finite discontinuous in $(c, c+2\pi)$

Continous function:

A function $f(x)$ is said to be continuous in an internal (a,b) if it is continuous at every Point of the interval (நம்ம எடுக்குற Point இந்த interval (a,b) la irontha athu Than Continous function)

Ex:

(i) $f(x) = \begin{cases} x & x < 1 \\ x^2 & x > 1 \end{cases}$



Dis Continuous

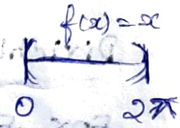
(ii) $f(x) = \begin{cases} x & x < 1 \\ x^2 & x \geq 1 \end{cases}$

but $x = 1$ is a Point of continity.

Continuous: Substitute the value Directly

Ex: $f(x) = x$ in $(0, 2\pi)$

$x = \pi$ is the point of continuity

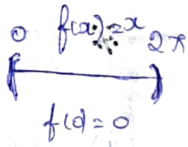


Discontinuous:

( ) → open [ ] → closed (serum)

End Point: Average value at the end point

Ex: $f(x) = x$ in $(0, 2\pi)$



Take $x = 0$ that in the end point

$$\text{Average} = \frac{f(0) + f(2\pi)}{2}$$

$f(x) = x$

$$= \frac{0 + 2\pi}{2}$$

$f(0) = 0$

$f(2\pi) = 2\pi$

$$= \pi$$

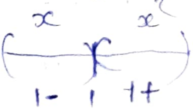
Middle Point: $\dfrac{LHL + RHL}{2}$

LHL – Left hand limit

RHL – Right hand limit

Ex:

$f(x) = \begin{cases} x & x < 1 \\ x^2 & x > 1 \end{cases}$



$x = 1$ in the point of discontinuity

$$\left.f(x) \text{ at } x=1\right\} = \frac{f(1^-) + f(1^+)}{2}$$

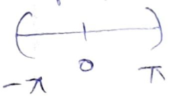
$$= \frac{1+1}{2} = 2/2 = 1$$

Problem

① Find the sum of fourier series of $f(x) = |x|$ in $-\pi < x < \pi$ at $x = 0$?

Given $f(x) = |x|$ in $(-\pi, \pi)$

$f(x) = |x|$



$x = 0$ is a point of continuity

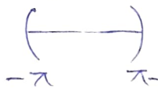
$$\left.f(x) \text{ at } x=0\right\} = f(0) = |0| = 0$$

Hence this is continous.

② Find the sum of fourier series of $f(x) = |x|$ in $-\pi < x < \pi$ at $x = \pi$?

Given $f(x) = |x|$ in $(-\pi, \pi)$

$f(x) = |x|$



$x = \pi$ is a point of discontinuity

$$\left.f(x) \text{ at } x=\pi\right] = \text{Average value of end point}$$

end point

$$= \frac{f(-\pi) + f(\pi)}{2}$$

$f(-\pi) = |-\pi| = \pi$

$$= \frac{\pi + \pi}{2}$$

$f(\pi) = |\pi| = \pi$

$$= \frac{2\pi}{2} = \pi$$

③ Find the sum of fourier series of

H.W $f(x) = |x|$ in $-\pi < x < \pi$ at $x = -\pi$?

Similar to above problem

Answer is π

Problem Under The Interval $(0, 2\pi)$

write The Formula for finding Euler's Constant of a fourier series in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x).\,dx \rightarrow ①$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\cos nx\,dx \rightarrow ②$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x).\sin nx\,dx \rightarrow ③$$

formula 1, 2, 3 are The Euler's formula

Parseval identity.

$$\frac{1}{\pi}\int_0^{2\pi} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

① Find The Fourier Series $f(x) = \frac{1}{2}(\pi - x)$ in the interval $(0, 2\pi)$. Hence deduce That $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{1}{4}$?

Given That $f(x) = \frac{1}{2}(\pi - x)$ in The interval $(0, 2\pi)$

To find The fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\,dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(2\pi^2 - \frac{4\pi^2}{2}\right) - (0-0)\right]$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2 - 4\pi^2}{2}\right]$$

$$= \frac{1}{2\pi}(0)$$

$$\boxed{a_0 = 0}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot \cos nx \cdot dx$$

Note: $\left\{\begin{matrix}\cos \\ \sin \\ e^x\end{matrix}\right\} \Rightarrow v$

Bernalli

d $\quad$ $\int$

$u = \pi - x \qquad v = \cos nx$

$u' = -1 \qquad v_1 = \frac{\sin nx}{n}$

$u'' = 0 \qquad v_2 = -\frac{\cos nx}{n^2}$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cdot \cos nx \cdot dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \overset{u}{(\pi - x)}\cdot \overset{v}{\cos nx}\cdot dx$$

$\sin 2n\pi = 0$

$\sin 0 = 0$

$$= \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2n\pi}{n^2} + \frac{\cos 0}{n^2}\right]$$

$\cos 2n\pi = 1$

$\cos 0 = 1$

$$= \frac{1}{2\pi}\left[-\frac{1}{n^2} + \frac{1}{n^2}\right] = 0$$

$$\boxed{a_n = 0}$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$$

$d \qquad \int$

$u = \pi - x \qquad v = \sin nx$

$u' = -1 \qquad v_1 = -\frac{\cos nx}{n}$

$u'' = 0 \qquad v_2 = -\frac{\sin nx}{n^2}$

$$= \frac{1}{2\pi}\left[-(\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{2\pi} \quad (\sin n x_0 = \sin 0)$$

$$= \frac{1}{2\pi}\left[\frac{(-\pi + x)\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi\cos 2n\pi}{n} + \frac{\pi\cos 0}{n}\right]$$

$\cos 2n\pi = 1$

$\cos 0 = 1$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right] \quad (\because 1 + 1 = 2)$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right]$$

$$\boxed{b_n = \frac{1}{n}}$$

W.K.T

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{0}{2} + \sum_{n=1}^{\infty} 0 \cdot \cos nx + \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

Deduce Part:

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Here by $f(x)$ we get $\sin nx$ so we put $x = \pi/2$

$x = \pi/2$ is a point of continuity $\quad 0 \quad \pi/2 \quad 2\pi$

$$\left.f(x)\text{ at } x = \pi/2\right\} = \sum_{n=1}^{\infty} \frac{1}{n}\sin \frac{n\pi}{2} \rightarrow \mathrm{I}$$

Here by given $f(x) = \frac{1}{2}(\pi - x)$

$$f(\pi/2) = \frac{1}{2}(\pi - \pi/2)$$

$$= \frac{1}{2}\left(\frac{2\pi - \pi}{2}\right)$$

$$\Rightarrow \boxed{f(\pi/2) = \pi/4}$$

Then by (I)

$$\pi/4 = \frac{1}{1}\sin\pi/2 + \frac{1}{2}\sin\frac{2\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin\frac{4\pi}{2} + \cdots$$

$$\pi/4 = \sin\pi/2 + \frac{1}{2}\sin\pi + \frac{1}{3}\sin 3\pi/2 + \frac{1}{4}\sin 2\pi + \cdots$$

$$\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Hence the proved

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(2) Find the Fourier series for the function $f(x) = (\pi - x)^2$ in $(0, 2\pi)$ and hence deduce that (i) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$ (ii) $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$

Given that $f(x) = (\pi - x^2)$ in the interval $(0, 2\pi)$

To find the fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cdot dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2\cdot dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(\pi - 2\pi)^3}{-3} - \frac{(\pi - 0)^3}{-3}\right] = \frac{1}{\pi}\left[\frac{-\pi^3}{-3} - \frac{\pi^3}{-3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cancel{-}\pi^3}{\cancel{-}3} - \frac{\pi^3}{-3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{2\pi^{\cancel{3}\,2}}{3}\right]$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

d	S
$u = (\pi - x)^2$	$v = \cos nx$
$u' = 2(\pi - x)$	$v_1 = \frac{\sin nx}{n}$
$= -2(\pi - x)$	
$u'' = -2(-1)$	$v_2 = -\frac{\cos nx}{n^2}$
$u'' = 2$	$v_3 = -\frac{\sin nx}{n^3}$
$u''' = 0$	

$$= \frac{1}{\pi}\int_0^{2\pi} \overset{u}{(\pi - x)^2}\, \overset{v}{\cos nx}\, dx$$

$$= \frac{1}{\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - 2(\pi - x)\left(\frac{\cos nx}{n^2}\right) - \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-2(\pi - 2\pi)\frac{\cos n2\pi}{n^2} + 2(\pi - 0)\left(\frac{\cos n0}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{+2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{4\cancel{\pi}}{n^2}\right]$$

$$\boxed{a_n = 4/n^2}$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx \, dx$$

d	∫
$u = (\pi - x)^2$ (+)	$v = \sin nx$
$u' = 2(\pi - x)$	$v_1 = -\frac{\cos nx}{n}$
$= -2(\pi - x)$ (−)	$v_2 = -\frac{\sin nx}{n^2}$
$u'' = 2$ (+)	$v_3 = +\frac{\cos nx}{n^2}$
$u''' = 0$ (−)	

$$= \frac{1}{\pi}\left[-(\pi - x)^2 \frac{\cos nx}{n} - 2(\pi - x)\frac{\sin nx}{2} + 2\frac{\cos nx}{n^3} \right]_0^{2\pi}$$

(upper limit − lower limit

wrong $= \frac{1}{\pi}\Big[-(\pi - 2\pi)^2 \frac{\cos n2\pi}{n}$ ① $+ (\pi - 0)^2 \frac{\cos 0}{n}$ ③ $+ 2\frac{\cos 2n\pi}{n^3}$ ② $- \frac{2\cos 0}{n^3}$ ④ $\Big]$

$$= \frac{1}{\pi}\left[\left(-(-\pi)^2 \cdot \frac{1}{n} + \frac{2}{n^3} \right) - \left(-\frac{\pi^2}{n} + \frac{2}{n^3} \right) \right]$$

$$= \frac{1}{\pi}\left[\left(\cancel{-\frac{\pi^2}{n}} + \cancel{\frac{2}{n^3}} \right) + \cancel{\frac{\pi^2}{n}} - \cancel{\frac{2}{n^3}} \right]$$

$$= \frac{1}{\pi}[0]$$

$$\boxed{b_n = 0}$$

W.K.T $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$f(x) = \frac{2\pi^2}{2 \times 3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx + 0$$

$$\therefore f(x) = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx \rightarrow \text{I}$$

i) Deduce Part:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

$(\pi-x)^2$

0 2π

If cos we use $x=0$ (or) $x=\pi$

if $x=0$ All terms +

if $x=l$ Alternative

+ - + +

$x=0$ Point of discontinuity

$$\left.\begin{array}{l} f(x) \\ \text{at } x=0 \end{array}\right\} = \frac{\pi^2}{3}+\sum_{n=1}^{\infty} 4/n^2$$

$$f(x)=(\pi-x)^2$$

$$f(0)=\pi^2$$

$$f(2\pi)=\pi^2$$

$$\frac{f(0)+f(2\pi)}{2} = \frac{\pi^2}{3}+\sum_{n=1}^{\infty} 4/n^2$$

$$\frac{\pi^2+\pi^2}{2} = \frac{\pi^2}{3}+4\sum_{n=1}^{\infty} 1/n^2$$

$$\frac{\cancel{2}\pi^2}{\cancel{2}} = \frac{\pi^2}{3}+4\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

$$\pi^2-\frac{\pi^2}{3} = 4\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

$$\frac{3\pi^2-\pi^2}{3\times4} = \left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

$$\frac{\cancel{2}\pi^2}{3\times\cancel{4}_2} = \left(1+\frac{1}{2^2}+\cdots\right)$$

$$\frac{\pi^2}{6} = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

(ii) Deduce part:

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots$$

Here by (I) $f(x)=\dfrac{\pi^2}{3}+\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$

If cos we use $x=0$ (or) $x=\pi$

By Parsval,

$$\frac{1}{\pi}\int_0^{2\pi}(f(x))^2dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

$$\frac{1}{\pi}\int_0^{2\pi}(\pi - x)^4 dx = \frac{\cancel{4}^{2}\pi^4}{\cancel{2}\times 9} + \sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\frac{1}{\pi}\left[\frac{(\pi-x)^5}{-5}\right]_0^{2\pi} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty} 1/n^4$$

$$\frac{1}{\pi}\left[\frac{+\pi^5}{+5} + \frac{\pi^5}{+5}\right] = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty} 1/n^4$$

$$\frac{1}{\cancel{\pi}}\left[\frac{2\pi^{\cancel{5}\,4}}{5}\right] = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty} 1/n^4$$

$$\left[\frac{2\pi^4}{5} - \frac{2\pi^4}{9}\right] = 16\sum_{n=1}^{\infty} 1/n^4$$

$$\frac{18\pi^4 - 10\pi^4}{45\times 16} = \sum_{n=1}^{\infty} 1/n^4$$

$$\frac{\cancel{8}\pi^4}{45\times \cancel{16}_{2}} = 1/1^4 + 1/2^4 + 1/3^4 + \cdots$$

$$\boxed{\frac{\pi^4}{90} = 1/1^4 + 1/2^4 + \cdots}$$

Problems under the interval $(0, 2l)$

Write the formula for finding Euler's constant of a fourier series in $(0, 2l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where

$$a_0 = 1/l \int_0^{2l} f(x)\,dx$$

$$a_n = 1/l \int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\cdot dx$$

Parseval Identity:

$$1/l \int_0^{2l} (f(x))^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Problem

① Find the Fourier series for the function $f(x) = (l-x)^2$ in $(0, 2l)$ and hence deduce that $\sum_{n=1}^{\infty} 1/n^2$

Given that $f(x) = (l-x)^2$ in the interval $(0, 2l)$

To find the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{\pi n x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

To find a_0:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2 dx.$$

$$= \frac{1}{l}\left[\frac{(l-x)^3}{-3}\right]_0^{2l}$$

$$= 1/l\left[\frac{(l-2l)^3}{-3} - \frac{(l-0)^3}{-3}\right]$$

$$= 1/l\left[\frac{(-l)^3}{-3} - \frac{(l)^3}{-3}\right]$$

$$= 1/l\left[\frac{\cancel{-}l^3}{\cancel{-}3} + \frac{(l^3)}{3}\right]$$

$$= \frac{1}{\cancel{l}}\left[\frac{2l^{\cancel{3}\,2}}{3}\right]$$

$$\boxed{a_0 = \frac{2l^2}{3}}$$

To find a_n:

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} \underset{u}{(l-x)^2}\ \underset{v}{\cos\frac{n\pi x}{l}}\,.dx$$

$d \qquad\qquad \int$

$u = (l-x)^2 \quad + \quad v = \cos\frac{n\pi x}{l}$

$u' = -2(l-x) \quad - \quad V_1 = \frac{\sin\frac{n\pi x}{l}}{n\pi/l}$

$u'' = +2 \quad + \quad V_2 = \frac{-\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}$

$u''' = 0 \qquad V_3 = \frac{-\sin\frac{n\pi x}{l}}{n^3\pi^3/l^3}$

$$a_n = 1/l\left[(l-x)^2 .\frac{\cancel{\sin\frac{n\pi x}{l}}}{n\pi/l} - \frac{2(l-x)\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2} - \frac{2\cancel{\sin\frac{n\pi x}{l}}}{n^3\pi^3/l^3}\right]$$

[$\sin 2n\pi = 0$
$\sin 0 = 0$]

$$= 1/l\left[\frac{-2(-l)(1)}{\frac{n^2\pi^2}{l^2}} + \frac{2(l)}{\frac{n^2\pi^2}{l^2}}\right]$$

$$= \frac{1}{\cancel{l}\times\frac{n^2\pi^2}{l^{\cancel{2}}}}[2l+2l] = \frac{l}{n^2\pi^2}[4l)$$

$$\therefore \boxed{a_n = \frac{4l^2}{n^2\pi^2}}$$

To find b_n:

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} \underbrace{(l-x)^2}_{u}\, \underbrace{\sin\frac{n\pi x}{l}}_{v}\,dx$$

d $\qquad\qquad$ $\int$

$u = (l-x)^2 \qquad v = \sin\frac{n\pi x}{l}$

$+$

$u' = -2(l-x) \qquad v_1 = \dfrac{-\cos\frac{n\pi x}{l}}{n\pi/l}$

$-$

$u'' = 2 \qquad v_2 = \dfrac{-\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}$

$+$

$u''' = 0 \qquad v_3 = \dfrac{-\cos n\pi x/l}{n^3\pi^3/l^3}$

$$= \frac{1}{l}\left[-(l-x)^2\frac{\cos\frac{n\pi x}{l}}{n\pi/l} - \cancel{2(l-x)\frac{\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}} + 2\frac{\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right]_0^{2l}$$

$\sin 2n\pi = 0$

$\sin 0 = 0$

$\cos 2n\pi = 1 \qquad \cos 0 = 1$

$$= \frac{1}{l}\left[\left(\frac{-(l^2)(1)}{n\pi/l} + \frac{2(1)}{\frac{n^3\pi^3}{l^3}}\right) - \left(-\frac{l^2(1)}{n\pi/l} + \frac{2(1)}{n^3\pi^3/l^3}\right)\right]$$

$$= \frac{1}{l}\left[\cancel{\frac{-l^2}{n\pi/l}} + \cancel{\frac{2}{\frac{n^3\pi^3}{l^3}}} + \cancel{\frac{l^2}{n\pi/l}} - \cancel{\frac{2}{n^3\pi^3/l^3}}\right]$$

$$= \frac{1}{l}[0] = 0$$

$$\therefore \boxed{b_n = 0}$$

W.K.T $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$f(x) = \frac{\cancel{2}l^2}{3\times\cancel{2}} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} 0$$

$$= \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cos\frac{n\pi x}{l}$$

Deduce Part:

$$\sum_{n=1}^{\infty} 1/n^2$$

If cos we use $x = 0$ (or) $x = l$

if $x = 0$ All terms +

if $x = l$ Alternative terms

Here all terms + so we use $x = 0$

$x = 0$ is a point of discontinity

$$\left.\begin{matrix} f(x) \\ x=0 \end{matrix}\right\} = \frac{l^3}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}$$

$$\Rightarrow \frac{f(0)+f(2l)}{2} = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$f(x) = (l-x)^2$$

$$f(0) = l^2 \text{ and } f(2l) = l^2$$

$$\Rightarrow \frac{l^2+l^2}{2} = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} 1/n^2$$

$$\Rightarrow \frac{\cancel{2}l^2}{\cancel{2}} - \frac{l^2}{3} = \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} 1/n^2$$

$$\Rightarrow \frac{3l^2 - l^2}{3} = \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} 1/n^2$$

$$\Rightarrow \frac{\cancel{2}l^2 \times \pi^2}{3 \times \cancel{4}_{2} l^2} = \sum_{n=1}^{\infty} 1/n^2$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty} 1/n^2$$

Hence The proved

---

(2) Find the fourier series for the function

sol $f(x) = 2x - x^2$ in the interval $0 < x < 3$

$(0, 2l)$ full range

$(0, l)$ 1/2 range

$(-l, l)$

Given that $f(x) = 2x - x^2$ in the interval $(0, 3)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

To find a_0:

Here $(0, 3)$
Method $(0, 2l)$
$\therefore 2l = 3$
$l = 3/2 \Rightarrow 1/l = 2/3$

$$a_0 = \frac{1}{l} \int_0^{2l} f(x)\, dx$$

$$= \frac{1}{l} \int_0^{2l} 2x - x^2\, dx$$

$$= \frac{1}{l} \left[\frac{2x^2}{2} - \frac{x^3}{3} \right]_0^{2l}$$

Here $\boxed{2l = 3}$, $\boxed{1/l = 2/3}$

$$= 2/3 \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3} \right]_0^3$$

$$= 2/3 \left[\left(9 - \frac{27}{3}\right) - (0 - 0) \right]$$

$$= 2/3 \left[\frac{27 - 27}{3} \right]$$

$$= 2/3\, (0)$$

$$\boxed{a_0 = 0}$$

To find a_n:

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{2l} \underset{u}{(2x-x^2)}\,\underset{v}{\cos\frac{n\pi x}{l}}\,dx$$

d		∫
$u = 2x - x^2$	$+$	$v = \cos\frac{n\pi x}{l}$
$u' = 2 - 2x$	$-$	$v_1 = +\frac{\sin\frac{n\pi x}{l}}{n\pi/l}$
$u'' = -2$	$+$	$v_2 = -\frac{\cos n\pi x/l}{n^2\pi^2/l^2}$
$u''' = 0$		$v_3 = -\frac{\sin\frac{n\pi x}{l}}{n^3\pi^3/l^3}$

$$= \frac{1}{l}\left[(2x-x^2)\frac{\sin\frac{n\pi x}{l}}{n\pi/l} + (2-2x)\frac{\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2} + 2\frac{\sin\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right]$$

$\sin 2n\pi = 0$

$\sin 0 = 0$

$$= \frac{1}{l}\left[\frac{(2-4l)(1)}{n^2\pi^2/l^2} - \frac{2(1)}{n^2\pi^2/l^2}\right]$$

$$= \frac{1}{\cancel{l}}\,\frac{1}{\frac{n^2\pi^2}{l^{\cancel{2}}}}\left[\cancel{2} - 4l - \cancel{2}\right]$$

$$= \frac{l}{n^2\pi^2}[-4l] = \frac{-4l^2}{n^2\pi^2}$$

Now $l = 3/2 \Rightarrow l^2 = 9/4$

$$\Rightarrow \frac{-4(9/4)}{n^2\pi^2} = \frac{-9}{n^2\pi^2} = a_n$$

$$\therefore \boxed{a_n = \frac{-9}{n^2\pi^2}}$$

To find bn

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\,\sin\frac{n\pi x}{l}\,.dx$$

$$b_n = \frac{1}{l}\int_0^{2l} (2x-x^2)\,\sin\frac{n\pi x}{l}\,.dx$$

d $\qquad\qquad$ $\int$

$u = 2x - x^2 \qquad v = \sin\frac{n\pi x}{l}$

$u' = 2 - 2x \qquad v_1 = \frac{-\cos\frac{n\pi x}{l}}{n\pi/l}$

$u'' = -2 \qquad v_2 = \frac{-\sin n\pi x/l}{n^2\pi^2/l^2}$

$u''' = 0 \qquad v_3 = \frac{\cos n\pi x/l}{n^3\pi^3/l^3}$

$$= \frac{1}{l}\left[-(2x-x^2)\frac{\cos\frac{n\pi x}{l}}{n\pi/l} + (2-2x)\frac{\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2} - \frac{2\cos\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\left(\frac{-(4l-4l^2)(1)}{n\pi/l} - \frac{2(1)}{n^3\pi^3/l^3}\right) - \left(0 - \frac{2(1)}{n^3\pi^3/l^3}\right)\right]$$

$$= \frac{1}{l}\left[\frac{-4l+4l^2}{n\pi/l} - \cancel{\frac{2}{n^3\pi^3/l^3}} + \cancel{\frac{2}{n^3\pi^3/l^3}}\right]$$

Now $l = 3/2$

$$= \frac{1}{l \times n\pi/l}\left[-4l+4l^2\right]$$

$$b_n = \frac{1}{n\pi}\left[-\cancel{4}^2\left(\frac{3}{\cancel{2}}\right) + \cancel{4}\left(\frac{9}{\cancel{4}}\right)\right]$$

$$= \frac{1}{n\pi}[-6+9]$$

$$\boxed{b_n = \frac{3}{n\pi}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{0}{2} + \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin\frac{n\pi x}{l}$$

Hence the proved

Fourier Series odd or even function $(-\pi,\pi)$ & $(-l,l)$

i) odd

$f(-x) = -f(x)$

Ex: $f(x) = x$

$f(-x) = -x$

$= -f(x)$

$\therefore f(x)$ is odd

$\therefore \boxed{a_0, a_n = 0}$

To find $b_n = ?$

ii) even

$f(-x) = f(x)$

Ex: $f(x) = x^2$

$f(-x) = (-x)^2$

$= x^2 = f(x)$

$\therefore f(x)$ is even

$\therefore \boxed{b_n = 0}$

To find $a_0, a_n = ?$

---

odd $\Rightarrow a_0, a_n = 0$

even $\Rightarrow b_n = 0$

1. Find the Fourier series for the function $f(x) = x^2$ is $(-\pi, \pi)$

Given that $f(x) = x^2$

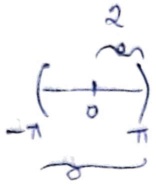
$\therefore f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore f(x)$ is a even function $\boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$



Full range ah Half range ah eluThanum na ~~xx~~ $\times 2$ apo full rang

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

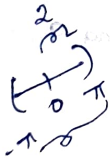
$$a_0 = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right]$$

$$= \frac{2}{\cancel{\pi}}\left[\frac{\pi^{\cancel{3}\,2}}{3}\right] = \frac{2\pi^2}{3}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$



$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$u = x^2$ $\quad + \quad$ $v = \cos nx$

$u' = 2x$ $\quad - \quad$ $v_1 = \dfrac{\sin nx}{n}$

$u'' = 2$ $\quad + \quad$ $v_2 = -\dfrac{\cos nx}{n^2}$

$u''' = 0$ $\qquad\quad$ $v_3 = \dfrac{-\sin nx}{n^3}$

$$= \frac{2}{\pi}\left[\cancel{\frac{x^2 \sin nx}{n}} + \frac{2x\cos nx}{n^2} - \cancel{\frac{2\sin nx}{n^3}}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{2\pi \cos n\pi}{n^2} - 0\right] \qquad \left[\because \cos n\pi = (-1)^n\right]$$

$$= \frac{4(-1)^n}{n^2}$$

$$\therefore \boxed{a_n = \frac{4(-1)^n}{n^2}}$$

Here $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

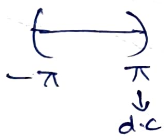
$$= \frac{2\pi^2}{\cancel{6}\,3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

$$\therefore f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \rightarrow \mathrm{I}$$

Deduce part:

$$1/1^2 + 1/2^2 + 1/3^2 + \cdots$$

Then by (I) $f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$



$x = \pi$ in (I) is a point of Discontinuity

$$\left.\begin{matrix} f(x) \text{ at} \\ x=\pi \end{matrix}\right\} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n(\pi)$$

$\cos n\pi = (-1)^n$

$$\frac{f(-\pi) + f(\pi)}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n \cdot (-1)^n}{n^2}$$

Here $f(x) = x^2$

$f(\pi) = \pi^2$

$f(-\pi) = (-\pi)^2 = \pi^2$

$$\frac{\pi^2+\pi^2}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\frac{\cancel{2}\pi^2}{\cancel{2}} - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} 1/n^2 \qquad \left[(-1)^{2n} = 1\right]$$

$$\frac{3\pi^2-\pi^2}{3\times 4} = \sum_{n=1}^{\infty} 1/n^2$$

$$\frac{\cancel{2}\pi^2}{3\times\cancel{4}\,2} = \left[1/1^2 + 1/2^2 + \dots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + 1/2^2 + \dots$$

---

(2) Find the fourier series for the function $f(x) = x$ in $(-l, l)$

$$f(x) = x$$

$$f(-x) = -x$$

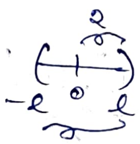
$$= -f(x)$$

$\therefore f(x)$ is odd $a_0 = 0$, $a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cdot dx$$

To find b_n:

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$



$$= \frac{2}{l}\int_{0}^{l} x \cdot \sin\frac{n\pi x}{l}\,dx$$

$u = x \quad + \quad v = \sin\frac{n\pi x}{l}$

$u' = 1 \quad - \quad v_1 = -\cos n\pi x/l \,/\, n\pi/l$

$u'' = 0 \quad + \quad v_2 = -\sin n\pi x/l \,/\, n^2\pi^2/l^2$

$$b_n = \frac{2}{l}\left[\frac{-x \cos \frac{n\pi x}{l}}{n\pi/l} + \cancel{\frac{\sin \frac{n\pi x}{l}}{n^2\pi^2/l^2}} \right]_0^l$$

$\sin n\pi = 0$

$$= \frac{2}{l}\left[\frac{-l \cos n\pi}{n\pi/l} - 0 \right]$$

$\cos n\pi = (-1)^n$

$$= \frac{-2l(-1)^n}{l \times n\pi/l}$$

$$\therefore \boxed{b_n = \frac{-2l(-1)^n}{n\pi}}$$

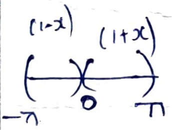
Hence $f(x) = \sum_{n=1}^{\infty} \frac{-2l(-1)^n}{n\pi} \sin \frac{n\pi x}{l}.$

---

③ Find the fourier series for the function $f(x) = \begin{cases} 1-x & -\pi < x < 0 \\ 1+x & 0 < x < \pi \end{cases}$ and $1/1^2 + 1/3^2 + 1/5^2$

Soln:

Given that $f(x) = \begin{cases} 1-x & -\pi < x < 0 \\ 1+x & 0 < x < \pi \end{cases}$



Here $f_1(x) = 1-x$

Put $x = -x$

$\Rightarrow f_1(-x) = 1+x = f_2(x)$

$f(x)$ is an even function so $\boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$f(x) = (1+x)$

$-\pi \quad 0 \quad \pi$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (1+x)\cdot dx$$

$$= \frac{2}{\pi}\left[x + \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\pi + \frac{\pi^2}{2}\right) - (0+0)\right]$$

$$= \frac{2}{\pi}\left(\pi + \frac{\pi^2}{2}\right)$$

$$= \frac{2\cancel{\pi}}{\cancel{\pi}} + \frac{\cancel{2}\pi^{\cancel{2}}}{\cancel{2}\cancel{\pi}}$$

$$\boxed{a_0 = 2 + \pi}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$f(x) = (1+x)$

$-\pi \quad 0 \quad \pi$

$$= \frac{2}{\pi}\int_0^{\pi} (1+x)\cos nx\, dx$$

d	$\int$
$u = 1+x$	$v = \cos n\pi$
$u' = 1$	$v_1 = \frac{\sin n\pi}{n}$
$u'' = 0$	$v_2 = -\frac{\cos n\pi}{n^2}$

$+ \quad -$

$$= \frac{2}{\pi}\left[(1+x)\frac{\cancel{\sin nx}}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$\{ (-1)^n \Rightarrow$ Put $n = 1, 2, 3, 4, \ldots$

$-2 \Rightarrow n = \text{odd}$

$0 \Rightarrow n = \text{even} \}$

$$a_n = \begin{cases} -2 \times \dfrac{2}{n^2\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases} = \begin{cases} \dfrac{-4}{n^2\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

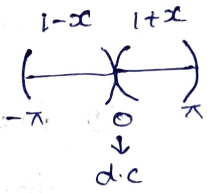
Then $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$

$$= \frac{2+\pi}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \frac{-4}{n^2\pi} \cos nx$$

$$f(x) = 1 + \pi/2 - \frac{4}{n^2\pi} \cdot \sum_{n=\text{odd}}^{\infty} 1/n^2 \cos nx .$$

$$f(x) = 1 + \pi/2 - \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} 1/n^2 \cos nx \rightarrow \text{I}$$

Deduce Part:-

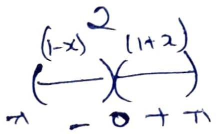


$1/1^2 + 1/3^2 + 1/5^2$

$\therefore x = 0$ point of discontinity

$$\left.\begin{matrix} f(x) \\ \text{at } x=0 \end{matrix}\right\} = 1 + \pi/2 - \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} 1/n^2 \qquad \cos 0 = 1$$

$$\frac{f(0-)+f(0+)}{2} = 1 + \pi/2 - 4/\pi \sum_{n=\text{odd}}^{\infty} 1/n^2$$



$f(0-) = 1 - 0 = 1$

$f(0+) = 1 + 0 = 1$

$$\frac{1+1}{2} = 1 + \pi/2 - \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} 1/n^2$$

$$\cancel{\pi}\cancel{-1} - \pi/2 = -\frac{4}{\pi} \sum_{n=odd}^{\infty} 1/n^2$$

$$-\pi/2 \times \pi/-4 = \sum_{n=odd}^{\infty} 1/n^2$$

$$\frac{\pi^2}{8} = \sum_{n=odd}^{\infty} 1/n^2 .$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Hence the proved

Half range Sine Series

First odd	even second
↓	↓
so a first Sine	Cosine so b second
↓	↓
$a_0, a_n = 0$	$b_n = 0$
$b_n = ?$	$a_0 = a_n = ?$

Formula $(-\pi, \pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

because of Half range we odd 2 $\leftarrow$ $\therefore b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$

① Find the half range Fourier Sine Series for the function $f(x) = x(\pi - x)$ in $(0, \pi)$ and hence deduce that $1/1^3 - 1/3^3 + 1/5^3 - \cdots$

$a_0 = 0$ and $a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

To find b_n:

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2) \sin nx \, dx$$

$d \qquad\qquad \int$

$u = \pi x - x^2 \qquad + \qquad v = \sin nx$

$u' = \pi - 2x \qquad - \qquad v_1 = \frac{-\cos nx}{n}$

$u'' = -2 \qquad + \qquad v_2 = \frac{-\sin nx}{n^2}$

$u''' = 0 \qquad\qquad v_3 = \frac{+\cos nx}{n^3}$

$$= \frac{2}{\pi}\left[-(\pi x - x^2)\frac{\cos nx}{n} + \cancel{(\pi - 2x)\frac{\sin nx}{n^2}} - \frac{2\cos nx}{n^3} \right]_0^{\pi}$$

$\sin n\pi = 0$

$\sin 0 = 0$

$$b_n = \frac{2}{\pi}\left[\left(0 - \frac{2\cos n\pi}{n^3}\right) - \left(0 - \frac{2\cos 0}{n^2}\right)\right]$$

$\cos n\pi = (-1)^n$

$$= \frac{2}{\pi}\left[\frac{-2}{n^3}(-1)^n + \frac{2}{n^3}\right]$$

$$= \frac{2}{\pi} \times \frac{2}{n^3}\left[-(-1)^n + 1\right]$$ → This concept written given below

$-(-1)+1 \to 2$

$-(-1)^2+1 = 0$

$$= \frac{4}{\pi n^3}\begin{cases} 2 & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$= \begin{cases} \frac{8}{n^3\pi} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$f(x) = \sum_{n=odd}^{\infty} b_n \sin nx = \sum_{n=odd}^{\infty} \frac{8}{n^3\pi} \sin nx$$

$$= 8/\pi \sum_{n=odd}^{\infty} \frac{1}{n^3} \sin nx$$

$$f(x) = \frac{8}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n^3} \sin nx$$

deduce part: $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} + \cdots$

Here we apply 0, or π we get 0

So

$0 \quad \pi/2 \quad \pi$

$x = \pi/2$ is a point of continuity

$$\underset{at\, x = \pi/2}{f(x)} = \frac{8}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n^3} \sin nx$$

$$f(\pi/2) = \frac{8}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n^3} \sin n\pi/2$$

$$\Rightarrow f(x) = \pi x - x^2$$

$$\Rightarrow f(\pi/2) = \pi(\pi/2) - (\pi/2)^2$$

$$= \frac{\pi^2}{2} - \frac{\pi^2}{4}$$

$$= \frac{4\pi^2 - 2\pi^2}{8} = \frac{\cancel{2}\pi^2}{\cancel{8}} = \frac{\pi^2}{4}$$

$$\frac{\pi^2}{4} = \frac{8}{\pi}\left[\frac{1}{1^3} \sin \pi/2 + \frac{1}{3^3} \sin \frac{3\pi}{2} + \cdots\right]$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$$

Hence the proved

* Half range Cosine Series:

Cosine
↓
even
↓
$b_n = 0$

$a_0, a_n = ?$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

① Find the Half range cosine series for the function $f(x) = x(\pi - x)$ in $(0, x)$ deduce that $1/1^4 + \frac{1}{2^4} + 1/3^4 + \ldots$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

To find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) - (0-0)\right]$$

$$a_0 = \frac{2}{\pi}\left[\frac{3\pi^3 - 2\pi^3}{6}\right]$$

$$= \frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{3}\,2}}{\cancel{6}\,3}\right] = \frac{\pi^2}{3}$$

$$\boxed{a_0 = \frac{\pi^2}{3}}$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\cos nx \, dx$$

d $\qquad\qquad$ $\int$

$u = \pi x - x^2 \quad + \quad v = \cos nx$

$u' = \pi - 2x \quad - \quad v_1 = \frac{\sin nx}{n}$

$u'' = -2 \quad + \quad v_2 = -\frac{\cos nx}{n^2}$

$u''' = 0 \qquad v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi}\left[\cancel{(\pi x - x^2)\frac{\sin nx}{n}} + (\pi - 2x)\frac{\cos nx}{n^2} + \cancel{\frac{2\sin nx}{n^3}}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi(-1)^n}{n^2} - \frac{\pi(1)}{n^2}\right]$$

$$= \frac{2}{\pi}\left(\frac{-\pi}{n^2}\right)\left[(-1)^n + 1\right] = -\frac{2}{n^2}\left[(-1)^n + 1\right]$$

odd

$(-1)^1 + 1 = -1+1 = 0$

$(-1)^3 + 1 = -1+1 = 0$

even

$(-1)^2 + 1 = 1+1 = 2$

$$= \begin{cases} 0 & n \text{ is odd} \\ -4/n^2 & n \text{ is even} \end{cases}$$

$$\therefore f(x) = \frac{\pi^2}{6} + \sum_{n=\text{even}}^{\infty} -4/n^2 \cos nx$$

To deduce part:

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

Half range $\Leftarrow$ $$\frac{2}{\pi}\int_0^{\pi} [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)^2\,dx = \frac{\pi^4}{9\times 2} + \sum_{n=\text{even}}^{\infty}\frac{16}{n^4}$$

$$\frac{2}{\pi}\int_0^{\pi}(\pi^2 x^2 + x^4 - 2\pi x^3).\,dx = \frac{\pi^4}{18} + 16\sum_{n=\text{even}} 1/n^4$$

$$\frac{2}{\pi}\left[\frac{\pi^2 x^3}{3} + \frac{x^5}{5} - \frac{2\pi x^4}{4}\right]_0^{\pi} = \frac{\pi^4}{18} + 16\left\{\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots\right\}$$

$$\frac{2}{\pi}\left[\frac{\pi^5}{3} + \frac{\pi^5}{5} - \frac{2\pi^5}{4}\right] = \frac{\pi^4}{18} + \cancel{\frac{16}{2^4}}\left\{\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right\}$$

$$\frac{2}{\pi}\left[\frac{10\pi^5 + 6\pi^5 - 15\pi^5}{30}\right] = \frac{\pi^4}{18} + \left\{\frac{1}{1^4} + \dots\right\}$$

$$\frac{\cancel{2}}{\cancel{\pi}}\left[\frac{\pi^{\cancel{5}\,4}}{\cancel{30}\,15}\right] - \frac{\pi^4}{18} = \frac{1}{1^4} + \frac{1}{2^4} + \dots\ \dots$$

$$\therefore \frac{\pi^4}{15} - \frac{\pi^4}{18} = \frac{1}{1^4} + \dots$$

$$\frac{6\pi^4 - 5\pi^4}{90} = \frac{1}{1^4} + \dots$$

$$\boxed{\frac{\pi^4}{90} = 1/1^4 + \dots}$$

(2) Complex form of fourier series:

Interval	$f(x)$	c_n
$(0, 2\pi)$	$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$	$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \cdot dx$
$(0, 2l)$	$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$	$c_n = \frac{1}{2l} \int_0^{2l} f(x) e^{-in\pi x/l} \cdot dx$
$(-\pi, \pi)$	$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$	$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \cdot dx$
$(-l, l)$	$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}$	$c_n = \frac{1}{2l} \int_0^{2l} f(x) e^{-in\pi x/l} \cdot dx$

① Find the complex form of fourier series for $f(x) = e^{ax}$ in $(-\pi, \pi)$

Soln:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

To find c_n:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax}\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax-inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x}\, dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{(a-in)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a-in)}\left[e^{(a-in)\pi} - e^{(a-in)(-\pi)}\right]$$

$e^{in\pi} = (-1)^n$

$e^{-in\pi} = (-1)^n$

$2\sinh x = e^{x} - e^{-x}$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi - in\pi} - e^{-a\pi + in\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}\cdot e^{-in\pi} - e^{-a\pi}\cdot e^{in\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}\cdot(-1)^n - e^{-a\pi}(-1)^n\right]$$

$$= \frac{(-1)^n}{2\pi(a-in)}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$C_n = \frac{(-1)^n}{2\pi(a-in)} 2\sinh a\pi$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{2\pi(a-in)} \cdot 2\sinh a\pi \cdot e^{inx}$$

(2) Find The Complex Form of Fourier series for $f(x) = e^{-x}$ in $(-1, 1)$

Soln:- $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$

To find C_n:-

$$C_n = \frac{1}{2l} \int_{-l}^{l} f(x) \cdot e^{in\pi x/l} \cdot dx$$

Here $l = 1$

$$C_n = \frac{1}{2} \int_{-1}^{1} e^{-x} \cdot e^{-in\pi x} \cdot dx$$

$$= \frac{1}{2} \int_{-1}^{1} e^{(-1-in\pi)x} \cdot dx$$

$$= \frac{1}{2} \left[\frac{e^{(-1-in\pi)x}}{(-1-in\pi)} \right]_{-1}^{1}$$

$$= \frac{1}{2(-1-in\pi)} \left[e^{(-1-in\pi)} - e^{(-1-in\pi)(-1)} \right]$$

$e^{in\pi} = (-1)^n$

$e^{-in\pi} = (-1)^n$

$e^{1} - e^{-1} = 2\sinh(1)$

$e^{x} - e^{-x} = 2\sinh x$

$$= \frac{1}{-2(1+in\pi)} \left[e^{-1} \cdot e^{-in\pi} - e^{1} e^{in\pi} \right]$$

$$= \frac{1}{-2(1+in\pi)} \left[e^{-1}(-1)^n - e^{1}(-1)^n \right]$$

$$= \frac{(-1)^n}{-2(1+in\pi)} \left[e^{-1} - e^{1} \right] = \frac{(-1)^n}{+2(1+in\pi)} \left[e^{1} - e^{-1} \right]$$

$$C_n = \frac{(-1)^n}{\cancel{2}(1+in\pi)} [\cancel{2}\sinh]$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(1+in\pi)} \sinh e^{in\pi x}$$

Harmonic Analysis:

The process of finding Euler constant for a tabular function is known as harmonic Analysis.

The Fourier constant are evaluated by the following Formulae: $\frac{2}{n} \Sigma y$

1) $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot dx$ (or) $a_0 = 2 \left[\frac{\Sigma f(x)}{n} \right]$

2) $a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\, dx$ (or) $a_n = 2 \left[\frac{\Sigma f(x) \cos nx}{n} \right]$

3) $b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\, dx$ (or) $b_n = 2 \left[\frac{\Sigma f(x) \sin nx}{n} \right]$

Fundamental or First Harmonic:

The term or $(a_1 \cos x + b_1 \sin x)$ in F.S is called First Harmonic

Second Harmonic:

The term or $(a_2 \cos 2x + b_2 \sin 2x)$ in F.S is called Second Harmonic and so on.

$a_0 = \frac{2}{n} \Sigma y$; $a_1 = \frac{2}{n} \Sigma y \cos x$; $a_2 = \frac{2}{n} \Sigma y \cos 2x$

$a_3 = \frac{2}{n} \Sigma y \cos 3x$

$b_1 = \frac{2}{n} \Sigma y \sin x$; $b_2 = \frac{2}{n} \Sigma y \sin 2x$; $b_3 = \frac{2}{n} \Sigma y \sin 3x$.

① Find the Fourier Series up to the Third Harmonic for $y = f(x)$ in $(0, 2\pi)$ defined by the table values given below

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	~~2π~~
y	1.0	1.4	1.9	1.7	1.5	1.2	~~1.0~~

Soln:

$\pi = 180$; $\pi/3 = \frac{180}{3} = 60$; $2\pi/3 = 2(60) = 120$

$4\pi/3 = 4(60) = 240$; $5\pi/3 = 5(60) = 300$

x	y	$y\cos x$	$y\cos 2x$	$y\cos 3x$	$y\sin x$	$y\sin 2x$	$y\sin 3x$
0	1.0	1	1	1	0	0	0
60	1.4	0.7	-0.7	-1.4	1.212	1.212	0
120	1.9	-0.95	-0.95	1.9	1.645	-1.645	0
180	1.7	-1.7	1.7	-1.7	0	0	0
240	1.5	-0.75	-0.75	1.5	-1.299	+1.299	0
300	1.2	0.6	-0.6	-1.2	-1.039	-1.039	0
	$\Sigma y =$ 8.7	$\Sigma y\cos x =$ -1.1	$\Sigma y\cos 2x =$ -0.3	$\Sigma y\cos 3x =$ 0.1	$\Sigma y\sin x =$ 0.519	$\Sigma y\sin 2x =$ -0.173	$\Sigma y\sin 3x =$ 0

$y\cos x$; $y\cos 2x$; $y\cos 3x$; $y\sin x$; $y\sin 2x$; $y\sin 3x$

$$a_0 = \frac{2}{n}\sum y = \frac{2}{6}(8.7)$$

$$= (0.33)(8.7) = 2.871$$

$$a_1 = \frac{2}{n}\sum y \cos x = \frac{2}{6}(-1.1)$$

$$= (0.33)(-1.1) = -0.363$$

$$a_2 = \frac{2}{n}\sum y \cos 2x = \frac{2}{6}(-0.3)$$

$$= (0.33)(-0.3) = -0.099$$

$$a_3 = \frac{2}{n}\sum y \cos 3x = \frac{2}{6}(0.1)$$

$$= (0.33)(0.1) = 0.033$$

$$b_1 = \frac{2}{n}\sum y \sin x = 0.173$$

$$b_2 = \frac{2}{n}\sum y \sin 2x = -0.057$$

$$b_3 = \frac{2}{n}\sum y \sin 3x = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x$$

$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

$$= \frac{2.871}{2} + (-0.363)\cos x + (-0.099)\cos 2x$$

$$+ (-0.099)\cos 3x + (0.173)\sin x$$

$$+ (0.057)\sin 2x$$

Type1: x in $(0, 2\pi)$

x : $0, \pi/3, 2\pi/3, 3\pi/3, \ldots, 2\pi$

$\pi = 180°$

x : $0, 60, 120, \ldots, 360$

Type: (2) Find an empirical formula of the form $f(x) = a_0 + a_1 \cos x + b_1 \sin x$ for the following data given and $f(x)$ in periodic with period 2π?

X	0	60	120	180	240	300	360
Y	40	31	-13.7	20	3.7	-21	40

x	y	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	40	40		0	
60	31	15.5		26.846	
120	-13.7	6.85		-11.864	
180	20	-20		0	
240	3.7	-1.85		-3.204	
300	-21	-10.5		18.186	

$\Sigma y = 60$ $\Sigma y \cos x = 30$ $\Sigma y \sin x = 29.964$

$a_0 = \frac{2}{n} \Sigma y = \frac{2}{\cancel{6}} \times \cancel{60}^{10} = 20$

$a_1 = \frac{2}{n} \Sigma y \cos x = \frac{2}{\cancel{6}_1} \times \cancel{30}^{5} = 10$

$b_1 = \frac{2}{n} \Sigma y \sin x = \frac{2}{6} \times 29.964 = 9.988$

$\therefore f(x) = \frac{20}{2} + 10 \cos x + 9.988 \sin x$

Type ③ The values of x and the corresponding values of $f(x)$ over a period T given below show that

$$f(x) = 0.75 + 0.37 \cos\theta + 1.004 \sin\theta$$

where $\theta = \frac{2\pi x}{T}$.

x	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
y	1.98	1.30	1.05	1.3	−0.88	−0.25	1.98

$x = 0 \Rightarrow \theta = \frac{2\pi}{T} \times (0) = 0$

$x = T/6 \Rightarrow \theta = \frac{2\pi}{\cancel{T}}\left(\frac{\cancel{T}}{\cancel{6}_3}\right) = \pi/3 = 60$

$x = T/3 \Rightarrow \theta = \frac{2\pi}{\cancel{T}}\left(\cancel{T}/3\right) = 2\pi/3 = 120$

$x = T/2 \Rightarrow \theta = \frac{2\pi}{\cancel{T}}\left(\cancel{T}/2\right) = \pi = 180$

$x = 2T/3 \Rightarrow \theta = \frac{2\pi}{\cancel{T}}\left(2\cancel{T}/3\right) = 4\pi/3 = 240$

$x = 5T/6 \Rightarrow \theta = \frac{2\pi}{\cancel{T}}\left(5\cancel{T}/6\right) = 10\pi/6 = 300$

x	θ	y	$y\cos\theta$	$y\cos 2\theta$	$y\sin\theta$	$y\sin 2\theta$
0	0	1.98	1.98		0	
$T/6$	60	1.30	0.65		1.1258	
$T/3$	120	1.05	−0.525		0.9093	
$T/2$	180	1.3	−1.3		0	
$2T/3$	240	−0.88	0.44		0.762	
$5T/6$	300	−0.25	−0.125		0.2165	
Σ		4.5	$\Sigma = 1.12$		$\Sigma = 3.0136$	

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$f(x) = \frac{1.5}{2} + 0.373 \cos\theta + 1.0045 \sin\theta,$$

where $\theta = \frac{2\pi x}{T}$

Unit - 3

Fourier Transform

Formula

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \cdot dx$$

Inversion formula

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F[f(x)]\, e^{-isx} \cdot ds$$

Parseval's Identity

$$\int_{-\infty}^{\infty} |f(x)|^2 \cdot dx = \int_{-\infty}^{\infty} |F f(x)|^2 ds$$

Problems:

① Show that the fourier transform of

15 $f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$ is $2\sqrt{\frac{2}{\pi}} \left(\frac{\sin as - as\cos as}{s^3} \right)$

Hence deduce that $\int_0^{\infty} \frac{\sin t - t\cos t}{t^3} dt = \frac{\pi}{4}.$

Also find the value of $\int_0^{\infty} \frac{(\sin t - t\cos t)^2}{t^6} \cdot dt$

Proof:

$\Rightarrow F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \cdot dx$

W.K.T $e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$ $\Big\} \Rightarrow$ De morgan's Theorm.

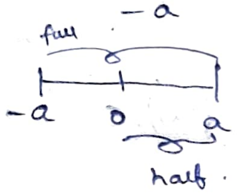
$\Rightarrow F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (a^2 - x^2)(\cos sx + i\sin sx) \cdot dx$

Then we have to apply the limit by given

$a^2 - x^2$ is $|x| < a \Rightarrow$ is written by $-a < x < a$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \underbrace{\int_{-a}^{a} (a^2 - x^2) \cos sx\, dx}_{\text{even fn}} + i \underbrace{\int_{-a}^{a} (a^2 - x^2) \sin sx \cdot dx}_{\text{odd fn so out}} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2) \cos sx\, dx$$



to change the limit

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a^2 - x^2) \cos sx\, dx$$

Now Bernoullis formula

$u = a^2 - x^2$ $\qquad dv = \cos sx\, dx$

$u' = -2x$ $\qquad v_1 = \frac{\sin sx}{s}$ $\qquad \rightarrow \int \sin nx\, dx = \frac{-\cos nx}{n}$

$u'' = -2$ $\qquad v_2 = -\frac{\cos sx}{s^2}$ $\qquad \rightarrow \int \cos nx\, dx = \frac{\sin nx}{n}$

$u''' = 0$ $\qquad v_3 = -\frac{\sin sx}{s^3}$

$$F[f(x)] = \frac{2}{\sqrt{2\pi}} \left\{ (a^2 - x^2) \frac{\sin sx}{s} - 2x \frac{\cos sx}{s^2} + 2 \frac{\sin sx}{s^3} \right\}_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \left[\left(\frac{-2a \cos sa}{s^2} + \frac{2 \sin sa}{s^3} \right) - \left(\frac{-2(0)}{s^2} + \frac{0}{s^3} \right) \right]$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{-a \cos as}{s^2} + \frac{\sin sa}{s^3} \right]$$ $\times$ by s^3

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{-as \cos as + \sin as}{s^3} \right]$$

$$\frac{2\times\sqrt{2}\times\sqrt{2}}{\sqrt{2\pi}}$$

$$F(f(x)) = 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)$$

Hence the first part

Deduce part :-

(i) $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \pi/4$ is Inversion formula because its very similar to 1st part

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(f(x))\, e^{-isx}\, ds$$

$$a^2 - x^2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}}\left(\frac{\sin as - as\cos as}{s^3}\right)(\cos sx - i\sin sx)\, ds$$

even function

$$= \frac{4}{\sqrt{2\pi}}\cdot\frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sx\, ds - i\int_{-\infty}^{\infty}\left(\frac{\sin as - as\cos as}{s^3}\right)\sin sx\, ds\right\}$$

$$a^2 - x^2 = \frac{4\times\cancel{2}}{\cancel{2}\pi}\int_0^\infty\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sx\, ds$$

$$a^2 - x^2 = \frac{4}{\pi}\int_0^\infty\left(\frac{\sin as - as\cos as}{s^3}\right)\cos sx\, ds$$

Put $a = 1$, $s = t$, $x = 0$

$$1^2 - 0^2 = \frac{4}{\pi}\int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt \quad (\cos 0 = 1)$$

$$\therefore \pi/4 = \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3} \right) dt$$

Hence the second part

Deduce part:

ii) $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3} \right)^2 . dt$ is Parseval's Identity

because its has power but similar to 1^{st} part.

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F f(x)|^2 ds$$

$|x| < a$

$-a < x < a$

$$\int_{-a}^{a} |a^2 - x^2|^2 dx = \int_{-\infty}^{\infty} \left| \frac{4}{\sqrt{2\pi}} \left(\frac{\sin as - as\cos as}{s^3} \right) \right|^2 . ds$$

By modulus we couldn't J or diff

$$\cancel{2} \int_0^a (a^2 - x^2)^2 dx = \cancel{2} \int_0^\infty \frac{16}{\cancel{2}\pi} \left(\frac{\sin as - as\cos as}{s^3} \right)^2 . ds$$

Put $a = 1$, $s = t$

$$2 \int_0^1 \underbrace{(1 - x^2)^2}_{(a-b)^2} dx = \frac{16}{\pi} \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3} \right)^2 . dt$$

$$2 \int_0^1 [1 + x^4 - 2x^2) dx = \frac{16}{\pi} \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3} \right)^2 . dt$$

$$2 \left(x + \frac{x^5}{5} - \frac{2x^3}{3} \right)_0^1 = \frac{16}{\pi} \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3} \right)^2 . dt$$

$$2 \left(\frac{1}{1} + \frac{1}{5} - \frac{2}{3} \right] = \frac{16}{\pi} \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3} \right)^2 . dt$$

$$2 \left(\frac{15 + 3 - 10}{15} \right) \frac{\pi}{16} = \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3} \right)^2 . dt$$

$$\pi/15 = \int_0^\infty \left(\frac{\sin t - t\cos t}{t^3} \right)^2 . dt$$

2) Find Fourier Transform of the

$$f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x| \geq 1 \end{cases}$$. Hence deduce

that $\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right) \cos\frac{s}{2}\, ds = \frac{3\pi}{16}$?

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

W.K.T

$$\left.\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}\right\} \Rightarrow \text{De Morgan Thm}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-x^2)\left(\cos sx + i \sin sx\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-1}^{1} (1-x^2) \cos sx\, dx + i \int_{-1}^{1} \sin sx\, dx \right.$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^1 (1-x^2) \cos sx\, dx$$

$$u = (1-x^2) \qquad \int dv = \int \cos sx\, dx$$

$$u' = -2x \qquad v_1 = \frac{\sin sx}{s}$$

$$u'' = -2 \qquad v_2 = \frac{-\cos sx}{s^2}$$

$$u''' = 0 \qquad v_3 = -\frac{\sin sx}{s^3}$$

$+$ $-$ $+$

$$F(f(x)) = \frac{2}{\sqrt{2\pi}}\left[\cancel{(1-x^2)\frac{\sin sx}{s}} - \frac{2x\cos sx}{s^2} + \frac{2\sin sx}{s^3}\right]_0^1$$

Cancel

$$= \frac{2}{\sqrt{2\pi}}\left[\frac{-2\cos s}{s^2} + \frac{2\sin s}{s^3}\right]$$

$$= \frac{2\times 2}{\sqrt{2\pi}}\left[\frac{-\cos s}{s^2} + \frac{\sin s}{s^3}\right]$$

$$= \frac{4}{\sqrt{2\pi}}\left(\frac{-s\cos s + \sin s}{s^3}\right)$$

$$F(f(x)) = \frac{4}{\sqrt{2\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)$$

Inversion Formula:.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(f(x))\, e^{-isx}\, ds$$

$$1-x^2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{4}{\sqrt{2\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)(\cos sx - i\sin sx)\,ds$$

$$= \frac{4}{2\pi}\left\{\underbrace{\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds}_{\text{even}} - i\underbrace{\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\sin sx\,ds}_{\text{odd}}\right.$$

$$1-x^2 = \frac{4}{\cancel{2}\pi}\times\cancel{2}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds$$

Put $x = 1/2$

$$1-\left(1/2\right)^2 = \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos s\,1/2\,ds$$

$$\therefore \int_0^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos s/2\,ds = 3\pi/16.$$

3) Find Fourier transform of the

$$f(x) = \begin{cases} a-|x|, & |x|<a \\ 0, & |x|>a \end{cases}$$ Hence deduce

That $$\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|)(\cos sx + i \sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a-|x|) \cos xs \cdot dx + i \int_{-a}^{a} (a-|x|) \sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^a (a-x) \cos sx \cdot dx$$

$u = a-x$ $\quad$ $v = \int \cos sx\, dx$

$u' = -1.$ $\quad$ $v_1 = \frac{\sin sx}{s}$

$u'' = 0$ $\quad$ $v_2 = -\frac{\cos sx}{s^2}$

$$F[f(x)] = \frac{2}{\sqrt{2\pi}} \left[(a-x)\frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \cdot -\frac{1}{s^2} \left[\cos sx \right]_0^a$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{-1}{s^2} [\cos sa - \cos 0]$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{-1}{s^2} [\cos sa - 1]$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{(1 - \cos as)}{s^2}$$

W.K.T. $1 - \cos 2\theta = 2\sin^2\theta$

$$1 - \cos\theta = 2\sin^2 \theta/2.$$

$$= \frac{2}{\sqrt{2\pi}} \frac{2\sin^2 as/2}{s^2}$$

$$= \frac{2}{\sqrt{2\pi}} \left(\frac{\sin as}{s}\right)^2$$

Inversion formula:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(f(x))\, e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \left(\frac{\sin \frac{as}{2}}{s}\right)^2 (\cos sx - i\sin sx) \, dx$$

$$= \frac{4}{2\pi} \left\{ \int_{-\infty}^{\infty} \left(\frac{\sin \frac{as}{2}}{s}\right)^2 \cos sx \, ds - i \int_{-\infty}^{\infty} \left(\frac{\sin as/2}{s}\right)^2 \sin sx \, ds \right.$$

$$a - |x| = \frac{4}{2\pi} \times 2 \int_0^{\infty} \left(\frac{\sin \frac{as}{2}}{s}\right)^2 \cos sx \, ds$$

$x \to 0, \; a \to 1$

$$1 = \frac{4}{\pi}\int_0^\infty \left(\frac{\sin s/2}{s}\right)^2 \cos 0 \, ds$$

$$\frac{\pi}{4} = \int_0^\infty \left(\frac{\sin s/2}{s}\right)^2 ds$$

$\Rightarrow s/2 = t \quad \Rightarrow s = 2t$

$\Rightarrow \frac{ds}{dt} = 2$

$\Rightarrow ds = 2dt$

$$\pi/4 = \int_0^\infty \left(\frac{\sin t}{2t}\right)^2 2dt$$

$$\pi/4 = \int_0^\infty \frac{(\sin t)^2}{2^{\cancel{2}} t^2} \cancel{2} dt$$

$$\pi/4 = \int_0^\infty \frac{(\sin t)^2}{2t^2} dt$$

$$2\pi/4 = \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$$

$$\pi/2 = \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt$$

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(4) Find the F.T of $f(x) = 1 - x, \ |x| \leq 1$

Hence deduce that ?

i) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \pi/3$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx) \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-1}^{1} (1-|x|) \cos sx\, dx + i \cancel{\int_{-1}^{1} (1-|x|) \sin sx\, dx} \right.$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{1} (1-x) \cos sx\, dx$$

$u = 1-x \qquad v = \cos sx\, dx$

$u' = -1 \qquad v_1 = \frac{\sin sx}{s}$

$u'' = 0 \qquad v_2 = -\frac{\cos sx}{s^2}$

$$= \frac{2}{\sqrt{2\pi}} \left((1-x) \frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right)_0^1$$

$$= \frac{2}{\sqrt{2\pi}} - \frac{1}{s^2} (\cos sx)_0^1$$

$$= \frac{2}{\sqrt{2\pi}} - \frac{1}{s^2} (\cos s - \cos 0)$$

$$= \frac{2}{\sqrt{2\pi}} - \frac{1}{s^2} [\cos s - 1] \qquad (\because 2\sin^2 \theta/2 = 1 - \cos\theta)$$

$$= \frac{2}{\sqrt{2\pi}} \frac{(1-\cos s)}{s^2}$$

$$F(f(x)) = \frac{2}{\sqrt{2\pi}} \left(\frac{2\sin^2 \theta/2}{s^2} \right) = \frac{4}{\sqrt{2\pi}} \left(\frac{\sin s/2}{s} \right)^2$$

Inversion formula:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(f(x))\, e^{-isx} \cdot ds$$

$$1-|x| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{4}{\sqrt{2\pi}} \left(\frac{\sin s/2}{s}\right)^2 (\cos sx - i \sin sx)\, ds$$

$$= \frac{4}{2\pi} \int_{-\infty}^{\infty} \left(\frac{\sin s/2}{s}\right)^2 \cos sx\, ds - i \int_{-\infty}^{\infty} \left(\frac{\sin s/2}{s}\right)^2 \sin sx$$

$$1-|x| = \frac{4}{2\pi} \times 2 \int_{0}^{\infty} \left(\frac{\sin s/2}{s}\right)^2 \cos sx\, ds$$

$x \to 0 \Rightarrow s/2 = t$

$\Rightarrow s = 2t$

$\Rightarrow \frac{ds}{dt} = 2 \Rightarrow ds = 2dt$

$$1-0 = \frac{4}{\pi} \int_{0}^{\infty} \left(\frac{\sin t}{2t}\right)^2 \cos 0 \cdot 2\, dt$$

$$1 = 4/\pi \int_{0}^{\infty} 1/4 \left(\frac{\sin t}{t}\right)^2 2\, dt$$

$$\pi/2 = \int_{0}^{\infty} \left(\frac{\sin t}{t}\right)^2 \cdot dt$$

Using Parsevals Identity.

$$\int_{-\infty}^{\infty} |f(x)|^2 \cdot dx = \int_{-\infty}^{\infty} |F f(x)|^2\, ds$$

$$\int_{-1}^{1} |1-|x||^2\, dx = \int_{-\infty}^{\infty} \left| \frac{4}{\sqrt{2\pi}} \left(\frac{\sin s/2}{s}\right)^2 \right|^2 ds$$

$$2\int_0^1 (1-x)^2\, dx = 2\int_0^\infty \frac{16}{2\pi}\left(\frac{\sin s/2}{s}\right)^4 . ds$$

$$2\int_0^1 (1^2+x^2-2x)\,dx = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s/2}{s}\right)^4 ds$$

$$2\left(x+\frac{x^3}{3}-\frac{2x^2}{2}\right)_0^1 = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin s/2}{s}\right)^4 . ds$$

$$s/2 = t \Rightarrow s = 2t \Rightarrow \frac{ds}{dt} = 2$$

$$\Rightarrow ds = 2dt$$

$$2\left(1+1/3-1\right) = \frac{16}{\pi}\int_0^\infty \left(\frac{\sin t}{2t}\right)^4 2dt$$

$$2\left(1/3\right) = \frac{\cancel{16}}{\pi}\int_0^\infty \frac{1}{\cancel{16}}\left(\frac{\sin t}{t}\right)^4 2dt$$

$$2/3 = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$$

$$\cancel{2}/3 \times \pi/\cancel{2} = \int_0^\infty \left(\frac{\sin t}{t}\right)^4 . dt$$

$$\pi/3 = \int_0^\infty \left(\frac{\sin t}{t}\right)^4 . dt$$

Hence the proved

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5) Find F.T of $f(x) = 1,\ |x| \leq a$ Hence deduce that

i) $\int_0^\infty \left(\frac{\sin t}{t}\right). dt = \pi/2$ ii) $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \pi/2$

$$F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (1)(\cos sx + i \sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (\cos xs)\,dx + i \int_{-a}^{a} \sin sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} \cos sx\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left(\frac{\sin sx}{s} \right)_0^a$$

$$F[f(x)] = \frac{2}{\sqrt{2\pi}} \left(\frac{\sin sa}{s} \right)$$

Inversion formula:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(f(x))\, e^{-isx}\, ds$$

$$1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2}{\sqrt{2\pi}} \left(\frac{\sin sa}{s} \right) (\cos sx - i \sin sx) ds$$

$$1 = \frac{2}{2\pi} \left\{ \int_{-\infty}^{\infty} \left(\frac{\sin sa}{s} \right) \cos sx\, dx - i \int_{-\infty}^{\infty} \left(\frac{\sin sa}{s} \right) \sin sx\, dx \right.$$

$$1 = \frac{1}{\pi} \cdot 2 \int_0^\infty \left(\frac{\sin sa}{s}\right) \cos sx \, ds$$

$x \to 0, \; a \to 1, \; s \to t$

$$1 = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right) \cdot \cos 0 \, dt$$

$$\pi/2 = \int_0^\infty \left(\frac{\sin t}{t}\right) \cdot dt$$

Using Parseval Identity:

$$\int_{-a}^{a} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F f(x)|^2 ds$$

$$\int_{-a}^{a} |1|^2 \cdot dx = \int_{-\infty}^{\infty} \left| \frac{2}{\sqrt{2\pi}} \left(\frac{\sin sa}{s}\right) \right|^2 \cdot ds$$

$$2 \int_0^a dx = 2 \int_0^\infty \frac{4}{2\pi} \left(\frac{\sin sa}{s}\right)^2 ds$$

$a \to 1, \; s \to t$

$$2 \int_0^1 dx = \frac{4}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^2 \cdot dt$$

$$2 (x)_0^1 = \frac{4}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^2 \cdot dt$$

$$\cancel{2}(1) \cdot \frac{\pi}{\cancel{4}_{2}} = \int_0^\infty \left(\frac{\sin t}{t}\right)^2 \cdot dt$$

$$\pi/2 = \int_0^\infty \left(\frac{\sin t}{t}\right)^2 . dt$$

Hence The proved

Sine and Cosine transform

1) write Fourier cosine transform pair:

The fourier cosine transform of $f(x)$ is defined as a

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= F_c(s)$$

The inverse Fourier Cosine transform of $F_c(s)$ is defined as

$$F_c^{-1}[F_c(s)] = f(x)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds.$$

2) write Fourier Sine transform pair

The fourier sine transform

of $f(x)$ is defined as,

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= F_s(s)$$

The inverse Fourier sine transform of $F_s(s)$ is defined as

$$F_s^{-1}[F_s(s)] = f(x)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \cdot \sin sx \, ds$$

Parseval's Identity For Single function:

Cosine transform for single function:

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_c(s)|^2 ds$$

Sine transform for single function:

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s(s)|^2 ds$$

Parseval's Identity for two function:

If $F_c(s)$ and $G_c(s)$ are the

fourier cosine transforms of

$f(x)$ and $g(x)$ respectively Then

$$\int_0^\infty f(x) \cdot g(x) \cdot dx = \int_0^\infty F_c(s)\, G_c(s)\, ds$$

If $F_s(s)$ and $G_s(s)$ are

fourier Cosine transforms of

$f(x)$ and $g(x)$ respectively Then

$$\int_0^\infty f(x) \cdot g(x) \cdot dx = \int_0^\infty F_s(s)\, G_s(s)\, ds$$

Problem

① Find the Fourier Cosine tranform of $f(x) = e^{-ax}$, $a \geq 0$ Then show that $\int_0^\infty \frac{\cos mx}{x^2+a^2}\, dx = \frac{\pi}{2a} e^{-ma}$

Given That $f(x) = e^{-ax}$

Forier Cosine transform

$$F_c(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$$

$$\therefore \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$$

$$\therefore \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

Now $a = -a$ and $b = s$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a \cos sx + s \sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2} (-a(1) + 0) \right]$$

$$F_c(e^{-ax}) = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+s^2} \right)$$

Next we have to find the Inverse fourier Cosine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(f(x)) \cos sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left(\sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+s^2} \right) \right) \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{a \cos sx}{a^2+s^2} \, ds$$

Put $x = m$

$$e^{-am} = \frac{2}{\pi}\int_0^\infty \frac{a \cos sm}{a^2+s^2}\, ds$$

Put $s = x$, $\frac{ds}{dx} = 1$, $ds = dx$

$$e^{-am} = \frac{2}{\pi}\int_0^\infty \frac{a \cos mx}{a^2+m^2}\, dx$$

$$\frac{\pi}{2a} e^{-am} = \int_0^\infty \frac{\cos mx}{a^2+m^2}\, dx$$

$$\therefore \int_0^\infty \frac{\cos mx}{a^2+m^2}\, dx = \frac{\pi}{2a} e^{-am}$$

Hence the proved

---

2) Find the Fourier Sine transform of $f(x) = e^{-ax}$, $a \geq 0$ show that

$$\int_0^\infty \frac{x \sin mx}{x^2+a^2}\, dx = \pi/2\, e^{-am}$$

Soln:

$$f(x) = e^{-ax}$$

Fourier Sine transform

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \sin sx\, dx$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$\therefore \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}$$

$$[a \sin bx - b \cos bx]$$

$$\therefore \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}$$

$$[a \cos bx + b \sin bx]$$

Put $a = -a$; $b = s$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{a^2+s^2} (-a \sin s - s \cos sx) \right]_0^\infty$$

$e^{\infty} = 0$

$e^{-\infty} = 0$

$e^{0} = 1$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{1}{a^2+s^2} (0 - (s)) \right]$$

$$F_s[e^{-ax}] = \sqrt{\frac{2}{\pi}} \left[\frac{s}{a^2+s^2} \right]$$

Next we have to find the Inverse

fourier Sine Transform

$$F(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(f(x)) \sin sx \, ds$$

$$e^{-ax} = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{s}{a^2+s^2} \right) \sin sx \, ds$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{a^2+s^2}\,ds$$

Put $x = m$

$$e^{-am} = \frac{2}{\pi}\int_0^\infty \frac{s\sin sm}{a^2+s^2}\,ds$$

Put $s = x$, $\frac{ds}{dx} = 1 \Rightarrow ds = dx$

$$e^{-am} = \frac{2}{\pi}\int_0^\infty \frac{x\sin mx}{a^2+x^2}\cdot dx$$

$$\pi/2\, e^{-am} = \int_0^\infty \frac{x\sin mx}{a^2+x^2}\cdot dx$$

$$\therefore \int_0^\infty \frac{x\sin mx}{a^2+x^2}\cdot dx = \frac{\pi}{2}e^{-am}$$

Hence The proved

Properties:

1) Find The Fourier Sine transform of $f(x)\sin ax$

Proof:

Given that The fourier sine

transform of $f(x) \sin ax$

Then $F_s[f(x)\sin ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)$

$\sin ax \sin sx \, dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\left[\cos(s-a)x - \cos (s+a)x\right] dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(s-a)x \, dx - \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(s+a)x \, dx\right]$$

$$= \frac{1}{2}\left[F_c(s-a) - F_s(s+a)\right]$$

$$F_s[f(x)\sin ax] = \frac{1}{2}\left[F_c(s-a) - F_s(s+a)\right]$$

---

2) Find the Fourier cosine transform of $f(x) \sin ax$.

Proof:

Given that the Fourier cosine transform of $f(x) \sin ax$

Then $F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}} \int_0^\infty$

$f(x)\sin ax \cos sx\, dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\left[\sin(s+a)x - \sin(s-a)x\, dx\right]$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(s+a)x\, dx - \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin(s-a)x\, dx\right]$$

$$= \frac{1}{2}\left[F_s(s+a) - F_s(s-a)\right]$$

$$\therefore F_c[f(x)\sin ax] = \frac{1}{2}\left[F_s(s+a) - F_s(s-a)\right]$$

Hence the proved.

---

3) Find the Fourier Sin transform of $f(xa)$

Proof:

Given that the Fourier Sin transform of $f(ax)$

$$F_s[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(ax) \sin sx \, dx$$

Here put $ax = t$

$$\Rightarrow x = t/a$$

$$\Rightarrow dx = \frac{dt}{a}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin s \, t/a \; dt/a$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin s/a \, t \, dt$$

$$= \frac{1}{a} \cdot F_s\left(\frac{s}{a}\right)$$

$$\therefore F_s[f(ax)] = \frac{1}{a} F_s\left(\frac{s}{a}\right)$$

Hence the proved.

Q Find the fourier cosine transform of

$$f(x) = \begin{cases} 1 - x^2, & 0 < x < 1 \\ 0, & \text{eleswhere} \end{cases}$$, Hence deduce

The value of $\int_0^{\infty} \frac{\sin x - x \cos x}{x^3} \cdot dx$

and $\int_0^{\infty} \frac{(x \cos x - \sin x)^2}{x^6} \cdot dx$

Soln:

Given $f(x) = \begin{cases} 1-x^2 & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 (1-x^2) \cos sx\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[(1-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(-\frac{\cos sx}{s^2}\right) + (-2)\left(-\frac{\sin sx}{s^3}\right) \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{2\cos s}{s^2} + \frac{2\sin s}{s^3} \right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{s^3} \right)$$

Now we have to find the inverse fourier transform

Taking inverse fourier transform

$$\sqrt{\frac{2}{\pi}} \int_0^\infty F_c(s) \cos sx \, ds = f(x)$$

$$\sqrt{\frac{2}{\pi}} \int_0^1 2\sqrt{\frac{2}{\pi}} \left(\frac{\sin s - s\cos s}{x^3} \right) \cos sx \, ds$$

$$= 1 - x^2$$

Now put $x = 0$, we have

$$\frac{4}{\pi} \int_0^1 \left(\frac{\sin s - s\cos s}{s^3} \right) ds = 1$$

$$\int_0^1 \left(\frac{\sin s - s\cos s}{s^3} \right) ds = \frac{\pi}{4}$$

Now put $s = x$, we have

$$\int_0^1 \frac{\sin x - x\cos x}{x^3} \, dx = \pi/4$$

Now we have to find the Parseval's Identity,

Using Parseval's Identity,

$$\int_{-\infty}^{\infty} |F(s)|^2 ds = \int_{-\infty}^{\infty} |f(x)|^2 . dx$$

$$\frac{8}{\pi}\int_0^1 \left(\frac{\sin s - s\cos s}{s^3}\right)^2 . ds = \int_0^1 (1-x^2)^2$$

$$= \int_0^1 (1-x^2)^2 . dx$$

$$= \int_0^1 (1-2x^2+x^4) . dx$$

$$= \left[x - 2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$= \left(1 - \frac{2(1)^3}{3} + \frac{(1)^5}{5}\right)$$

$$= \left(1 - \frac{2}{3} + 1/5\right)$$

$$= \frac{8}{15}$$

$$\therefore \int_0^1 \frac{(\sin s - s\cos s)^2}{s^6} ds = \frac{\pi}{15}$$

Put $s = x$

$$\int_0^1 \frac{(\sin x - x\cos x)^2}{x^6} \cdot dx = \frac{\pi}{15}$$

Hence the proved.

---

(2) Find the fourier Sine transform of $\frac{x}{x^2+a^2}$ and fourier cosine transform of $\frac{a}{x^2+a^2}$.

Soln:-

Let $f(x) = e^{-ax}$

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cdot \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-ax}}{s^2+a^2} (-a\cos sx + s\sin sx) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \frac{a}{s^2+a^2}$$

Now we have to find the inverse Cosine fourier transform.

Taking inverse cosine fourier transform.

$$\sqrt{\frac{2}{\pi}}\int_0^\infty F_c(s)\cos sx\,ds = f(x)$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{a}{s^2+a^2}\cos sx\,ds = e^{-ax}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{a}{s^2+a^2}\cos sx\,ds = \sqrt{\frac{\pi}{2}}\,e^{-ax}$$

Now put $s = x$ and $x = s$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{a}{x^2+a^2}\cos sx\,dx = \sqrt{\pi/2}\,e^{-as}$$

$$F_c\left(\frac{a}{x^2+a^2}\right) = \sqrt{\pi/2}\,e^{-as}$$

$$F_s[f(x)] = \sqrt{2/\pi}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s[e^{-ax}] = \sqrt{2/\pi}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{s^2+a^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{s}{x^2+a^2}$$

Taking inverse sine fourier transform

$$\sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds = f(x)$$

$$\sqrt{2/\pi} \int_0^\infty \sqrt{2/\pi} \frac{s}{s^2+a^2} \sin sx \, ds = e^{-ax}$$

$$\sqrt{2/\pi} \int_0^\infty \frac{s}{a^2+s^2} \sin sx \, ds = \sqrt{\pi/2}\, e^{-ax} \rightarrow (1)$$

T.P :- $$F_s\left(\frac{x}{x^2+a^2}\right) = \sqrt{\pi/2}\, e^{-ax}$$

Put $s = x$ and $x = s$ in (1)

$$\sqrt{2/\pi} \int_0^\infty \frac{x}{x^2+a^2} \sin sx \, dx = \sqrt{\pi/2}\, e^{-ax}$$

$$F_s\left(\frac{x}{x^2+a^2}\right) = \sqrt{\pi/2}\, e^{-as}$$

Hence the Proved.

Unit - 4

Classification of PDE :

i) Elliptic $\Rightarrow B^2-4AC<0$

ii) Parabolic $\Rightarrow B^2-4AC=0$

iii) Hyperbolic $\Rightarrow B^2-4AC>0$

① Classify the PDE $\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\partial y}+4\frac{\partial^2 f}{\partial y^2}=0$

$$f_{xx}+2f_{xy}+4f_{yy}=0$$

$$A=1,\ B=2,\ C=4$$

$$\Rightarrow B^2-4AC \Rightarrow (2)^2-4(1)(4)$$

$$\Rightarrow 4-16$$

$$\Rightarrow -12<0$$

$$\therefore B^2-4AC<0$$

Then the given PDE is elliptic.

② Classify the PDE $u_{xx}+xu_{yy}=0$

Here $A=1,\ B=0,\ C=x$

$$\Rightarrow B^2-4AC \Rightarrow (0)^2-4(1)(x) \Rightarrow -4x$$

$$B^2-4AC=-4x$$

If $x=0 \Rightarrow B^2-4AC=-4\overset{(0)}{x}=0 \Rightarrow$ Parabolic

If $x<0 \Rightarrow B^2-4AC=-4x>0 \Rightarrow$ Hyperbolic

If $x>0 \Rightarrow B^2-4AC<0 \Rightarrow$ elliptic

(3) Classify the PDE $f_{xx} + 2f_{xy} + f_{yy} - 3f_x - 4f_y + 10 = 0$

Here $A = 1,\ B = 2,\ C = 1$

$\Rightarrow B^2 - 4AC \Rightarrow (2)^2 - 4(1)(1)$

$\Rightarrow 4 - 4 = 0$

$\therefore B^2 - 4AC = 0.$

Then the given PDE is Parabolic.

(4) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

$A = 1,\ B = 0,\ C = -1$

$B^2 - 4AC > 0$

Equation is Hyperbolic

(5) $$\frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial y}\right) + xy$$

$A = 0,\ B = 1,\ C = 0$

$B^2 - 4AC = 1 - 0 > 0$

Equation is hyperbolic.

Applications of Partial Differential eqn:

1. One Dimensional wave equation - String
2. One Dimensional heat equation - Rod
3. 2-D heat equation - plate

1-D wave equation:. (V.N.G)

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

① Write the Possible solution of 1-D wave eqn:.

2 mark

$$y(x,t) = (Ae^{px} + Be^{-px})(ce^{pat} + De^{-pat})$$

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$$

$$y(x,t) = (Ax + B)(ct + D)$$

Boundary Condition:.

① $y(0,t) = 0 \quad \forall t$

② $y(l,t) = 0 \quad \forall t$

③ $\frac{\partial}{\partial t} y(x,0) = 0$ (Initial velocity)

④ $y(x,0) = f(x)$ (Displacement)

① A String is stretched and fastened to two points $x=0$ and $x=l$ apart, Motion is started by displacing the String into the form $y = k(lx - x^2)$ from which it is released at time $t=0$. Find the displacement of any Point on the String at a distance at x From one end at time t?

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary condition.

i) $y(0,t) = 0 \quad \forall t$

ii) $y(l,t) = 0 \quad \forall t$

iii) $\frac{\partial y}{\partial t}(x,0) = 0$ ($\because$ Initial velocity is zero)

iv) $y(x,0) = k(lx - x^2)$

The Correct Solution which satisfy our boundary Condition is given b

$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$ → I

(i) by I eqn we put $y(0,t) = 0$;

$\sin n\pi = 0$
$\sin 0 = 0$
$\cos 0 = 1$
$\cos n\pi = (-1)^n$

$y(0,t) = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$

$0 = (A(1) + B(0))(C\cos pat + D\sin pat)$

$0 = A(C\cos pat + D\sin pat)$

$(x+3)(x+4) = 0$
$x = 0,\ x+3 = 0,\ x+4 = 0$

Then $A = 0$ and $(C\cos pat + D\sin pat) \neq 0$

Then I eqn be $y(x,t) = B\sin px(C\cos pat + D\sin pat)$ → II

ii) by II eqn we put $y(l,t) = 0$;

$y(l,t) = B\sin pl(C\cos pat + D\sin pat)$

$0 = B \cdot \sin pl(C\cos pat + D\sin pat)$

$B \neq 0$, $\sin pl = 0$ and $(C\cos pat + D\sin pat) \neq 0$

Here $\sin n\pi = 0 \quad \therefore \cancel{\sin} n\pi = \cancel{\sin} pl$

$\therefore \boxed{p = n\pi/l}$

Then II eqn be $y(x,t) = B\sin \frac{n\pi}{l}x\left(C\cos \frac{n\pi}{l}at + D\sin \frac{n\pi}{l}at\right)$

$\left[\frac{d}{dx}(\sin nx) = n\cos x\right]\ \left[\frac{d}{dx}(\cos nx) = n - \sin x\right]$

$$\frac{\partial y}{\partial t}(x,t) = B \sin\frac{n\pi}{l}x \left(-C \sin n\pi/l\, at\; n\pi/l\, a + D\cos n\pi/l\, at\; n\pi/l\, a\right)$$

iii) $\frac{\partial y}{\partial t}(x,0) = 0 \rightarrow$

$$0 = B\sin\frac{n\pi}{l}x\left(-C\, n\pi/l\, a \sin 0 + D\, n\pi/l\, a \cos 0\right)$$

$$0 = B\sin\frac{n\pi}{l}x\; D\, n\pi/l\, a \qquad (\because \cos 0 = 1)$$

$B \neq 0$; $\sin\frac{n\pi}{l}x \neq 0$; $D = 0$; $n\pi/l\, a \neq 0$

$$y(x,t) = B\sin\frac{n\pi}{l}x\left(C\cos n\pi/l\, at\right)$$

$$= BC \sin n\pi/l\, x \cos n\pi/l\, at$$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi/l\, x \cdot \cos n\pi/l\, at \rightarrow \text{IV}$$

iv) $y(x,0) = k(lx - x^2) \Rightarrow$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin n\pi/l\, x$$

$$k(lx - x^2) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}x \cdot dx$$

$$= \frac{2}{l}\int_0^l k(lx - x^2)\sin\frac{n\pi}{l}x \cdot dx$$

$$= \frac{2k}{l}\int_0^l (lx - x^2)\sin\frac{n\pi}{l}x \cdot dx$$

$\int \sin nx\,dx = \frac{-\cos nx}{n}$

$\int \cos nx\,dx = \frac{\sin nx}{n}$

$u = (lx - x^2)$ (+) $v = \sin n\pi/l\, x$

$u' = (l - 2x)$ (−) $v_1 = -\cos n\pi/l\, x / n\pi/l$

$u'' = -2$ (+) $v_2 = -\sin n\pi/l\, x / n^2\pi^2/l^2$

$u''' = 0$ $v_3 = \cos n\pi/l\, x / (n\pi/l)^3$

$$b_n = \frac{2k}{l}\left(\frac{-(lx-x^2)\cdot\cos\frac{n\pi x}{l}}{n\pi/l} + \frac{(l-2x)\sin\frac{n\pi x}{l}}{(n\pi/l)^2} - \frac{2\cos\frac{n\pi}{l}x}{(n\pi/l)^3}\right)_0^l$$

$\sin 0 = 0$

$\sin n\pi = 0$

$\cos 0 = 1$

$\cos n\pi = (-1)^n$

$$= \frac{2k}{l}\left(\frac{-2\cos\frac{n\pi}{l}x}{(n\pi/l)^3}\right)_0^l$$

$$= \frac{2k}{l} \times \frac{-2}{(n\pi/l)^3}\left(\cos n\pi/l\, x\right)_0^l$$

$$= \frac{-4k}{l \times \frac{n^3\pi^3}{l^{3\,2}}}\left(\cos n\pi/l\, l - \cos n\frac{\pi}{l}0\right)$$

$$= \frac{-4k}{\frac{n^3\pi^3}{l^2}}\left(\cos n\pi - \cos 0\right)$$

$$b_n = \frac{4kl^2}{n^3\pi^3}\left(1 - \cos n\pi\right)$$

$$b_n = \frac{4kl^2}{n^3\pi^3}\left(1 - (-1)^n\right)$$

$$y(x,t) = \sum_{n=1}^{\infty} \frac{4kl^2}{n^3\pi^3}\left(1-(-1)^n\right)\sin\frac{n\pi}{l}x \cdot \cos\frac{n\pi}{l}at$$

(2) A String is stretched and fastended to two Points $x=0$ and $x=l$, apart, Motion is started by displacing the string into the form $y = y_0 \sin^3 \pi x/l$ from which it is releas

at time $t=0$. Find the displacement of any Point on the String at a distance at x from one end at time t?

The wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$

From the given Problem, we get the following boundary Condition.

i) $y(0,t)=0$ (ii) $y(l,t)=0$ (iii) $\frac{\partial y}{\partial t}(x,0)=0$

iv) $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi/l \, x \cdot \cos n\pi/l \, at \rightarrow (IV)$$

iv) $y(x,0) = y_0 \sin^3 \pi x/l \Rightarrow$

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cdot \cos 0$$

$$y_0 \sin^3 \pi x/l = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

$$\left[\text{w.k.T} \quad \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}\right]$$

$$y_0 \left(\frac{3 \sin \frac{\pi x}{l} - \sin 3 \frac{\pi x}{l}}{4} \right) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

$$\frac{3y_0}{4} \sin \frac{\pi x}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} = b_1 \sin \frac{\pi}{l} x + b_2 \sin \frac{2\pi}{l} x + b_3 \sin \frac{3\pi}{l} x.$$

$$b_1 = \frac{3y_0}{4}; \quad b_2 = 0; \quad b_3 = -\frac{y_0}{4}$$

by eqn IV $\Rightarrow y(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi/l \, x \cdot \cos n\pi/l \, at$

$$y(x,t) = b_1 \sin \pi/l \, x \cdot \cos \pi/l \, at + b_3 \sin 3\pi/l \, x \cdot \cos 3\pi/l \, at$$

$$y(x,t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cdot \cos \pi/l \, at - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos 3\pi/l \, at$$

Type 2: velocity

i) $y(0,t)=0$

ii) $y(l,t)=0$

iii) $y(x,0)=0$

iv) $\frac{\partial y}{\partial t}(x,0)=f(x)$

$y(x,t)=(A\cos px + B\sin px)(C\cos pat + D\sin pat)$

① A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity given by

$\frac{\partial y}{\partial t}$ - velocity

$$v=\begin{cases} cx & \text{if } 0\le x\le l/2 \\ c(l-x) & \text{if } l/2\le x\le l \end{cases}$$

Find the displacement function $y(x,t)$

The wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$

From the given problem, we get the following boundary conditions

i) $y(0,t)=0$

ii) $y(l,t)=0$

iii) $y(x,0)=0$

iv) $\frac{\partial y}{\partial t}(x,0)=\begin{cases} cx & \text{if } 0\le x\le l/2 \\ c(l-x) & \text{if } l/2\le x\le l \end{cases}$

The correct solution which satisfies our boundary condition is given by

$y(x,t)=(A\cos px + B\sin px)(C\cos pat + D\sin pat)$ →

i) $y(0,t) = 0 \Rightarrow$

$$y(0,t) = (A\cos 0 + B\sin 0)(c\cos pat + D\sin pat)$$

$$0 = A(c\cos pat + D\sin pat)$$

$$A = 0 \text{ and } c\cos pat + D\sin pat \neq 0$$

$$\text{I} \Rightarrow y(x,t) = B\sin px\,(c\cos pat + D\sin pat) \rightarrow \text{II}$$

ii) by II eqn we put $y(l,t) = 0$;

$$y(l,t) = B\sin pl\,(c\cos pat + D\sin pat)$$

$$B \neq 0;\ \sin pl = 0 \text{ and } c\cos pat + D\sin pat \neq 0$$

Here $\sin n\pi = 0 \quad \therefore \sin pl = \sin n\pi$

$$\boxed{p = n\pi/l}$$

Then II eqn be $y(x,t) = B\cdot\sin\left(\frac{n\pi}{l}\right)x\left(c\cos\left(\frac{n\pi}{l}\right)at + D\sin\left(\frac{n\pi}{l}\right)at\right) \rightarrow \text{III}$

(iii) $y(x,0) = 0$

$$y(x,0) = B\sin\left(\frac{n\pi}{l}\right)x\,(c\cos 0 + D\sin 0)$$

$$0 = B\sin\left(\frac{n\pi}{l}\right)x\,(c)$$

$$B \neq 0;\ \sin\frac{n\pi}{l}x \neq 0;\ c = 0$$

Then III eqn be $y(x,t) = B\sin\left(\frac{n\pi x}{l}\right)\cdot\left(D\sin\frac{n\pi}{l}at\right)$

$$\therefore y(x,t) = B \times D \times \sin\left(\frac{n\pi x}{l}\right)\cdot\sin\left(\frac{n\pi}{l}at\right)$$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{l}x \cdot \sin\frac{n\pi}{l}at \rightarrow \text{IV}$$

't' diff

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{l}x \cdot \cos\frac{n\pi}{l}at\left(\frac{n\pi}{l}a\right)$$

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{l}x \cdot \cos 0\left(\frac{n\pi}{l}a\right)$$

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} \left(C_n \left(\frac{n\pi}{l} a \right) \right) \sin \frac{n\pi}{l} x$$

$$\left[\therefore b_n = C_n \frac{n\pi}{l} a \right]$$

$$\begin{cases} cx & \text{if } 0 \le x \le l/2 \\ c(l-x) & \text{if } l/2 \le x \le l \end{cases} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$$

$$= \frac{2}{l} \left\{ \int_0^{l/2} cx \sin \frac{n\pi}{l} x \, dx + \int_{l/2}^{l} c(l-x) \sin \frac{n\pi}{l} x \right.$$

$$= \frac{2c}{l} \left\{ \int_0^{l/2} x \sin \frac{n\pi x}{l} \, dx + \int_{l/2}^{l} (l-x) \sin \frac{n\pi x}{l} \, dx \right.$$

$\int u \, dv = uv_1 - u'v_2 + u''v_3$

$u = x \qquad v = \sin \frac{n\pi x}{l} \qquad u = l - x$

$u' = 1 \qquad v_1 = -\frac{\cos \frac{n\pi}{l} x}{n\pi/l} \qquad u' = -1$

$u'' = 0 \qquad v_2 = -\frac{\sin \frac{n\pi}{l} x}{(n\pi/l)^2} \qquad (-1) \qquad u'' = 0$

$$b_n = \frac{2c}{l} \left\{ \left(-\frac{x \cos \frac{n\pi}{l} x}{n\pi/l} + \frac{\sin \frac{n\pi}{l} x}{n\pi} \right)_0^{\frac{l}{2}} + \right.$$

$$\left(\frac{-(l-x) \cos \frac{n\pi}{l} x}{n\pi/l} - \frac{\sin \frac{n\pi}{l} x}{(n\pi/l)^2} \right)_{l/2}^{l}$$

$$= \frac{2c}{l}\left\{\left[\left(\frac{-\frac{l}{2}\cos\left(\frac{n\pi}{l}\frac{l}{2}\right)}{n\pi/l} + \frac{\sin\left(\frac{n\pi}{l}\frac{l}{2}\right)}{(n\pi/l)^2}\right) - \left(\frac{-0}{n\pi/x} + \frac{\sin 0}{(n\pi/l)^2}\right)\right] + \right.$$

$$\left.\left[\left(\frac{-(l-l)\cos\left(\frac{n\pi}{l}l\right)}{n\pi/l} - \frac{\sin\left(\frac{n\pi}{l}l\right)}{(n\pi/l)^2}\right) - \left(\frac{+\left(l+\frac{l}{2}\right)\cos\left(\frac{n\pi}{l}\frac{l}{2}\right)}{n\pi/l} - \frac{\sin\left(\frac{n\pi}{l}\frac{l}{2}\right)}{(n\pi/l)^2}\right)\right]\right\}$$

$\sin 0 = 0 \quad \cos 0 = 1$

$\sin n\pi = 0 \quad \cos n\pi = (-1)^n$

$$= \frac{2c}{l}\left\{\frac{-\frac{l}{2}\cos n\pi/2}{n\pi/l} + \frac{\sin\left(\frac{n\pi}{l}\frac{l}{2}\right)}{(n\pi/l)^2} + \frac{\frac{l}{2}\cos n\pi/2}{n\pi/l} + \frac{\sin n\pi/2}{(n\pi/l)^2}\right\}$$

$$b_n = \frac{2c}{l}\left\{\frac{2\sin n\pi/2}{(n\pi/l)^2}\right\}$$

$$b_n = \frac{4c}{l\left(\frac{n^2\pi^2}{l^2}\right)} \cdot \sin n\pi/2 = \frac{4cl}{n^2\pi^2} \cdot \sin n\pi/2.$$

$$(b_n) = \frac{4cl}{n^2\pi^2} \cdot \sin n\pi/2$$

$$C_n \frac{n\pi}{l} a = \frac{4cl}{n^2\pi^2} \sin n\pi/2$$

$$\boxed{C_n = \frac{4cl^2}{n^3\pi^3 a} \cdot \sin\frac{n\pi}{2}}$$

$$\text{IV} \Rightarrow y(x,t) = \sum_{n=1}^{\infty} C_n \cdot \sin\frac{n\pi}{l}x \cdot \sin\frac{n\pi}{l}at$$

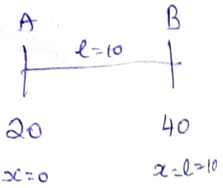
$$y(x,t) = \sum_{n=1}^{\infty} \frac{4cl^2}{(n\pi)^3 a} \cdot \sin n\pi/2 \; \sin\frac{n\pi}{l}x \cdot \sin\frac{n\pi}{l}$$

<u>Heat Equation:</u>

(2) A bar 10 cm long with insulated sides, has its ends A and B kept at 20°c and 40°c respectively, until steady state conditions prevails. The temperature at A is then raised to 50°c and B is lowered to 10°c. Find the temperature $u(x,t)$

The steady state temperature is

$$u = \left(\frac{B-A}{l}\right)x + A$$

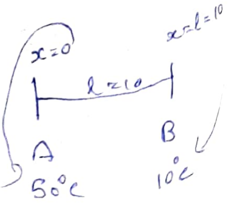


$$= \left(\frac{40-20}{10}\right)x + 20$$

$$\boxed{u = 2x + 20}$$

After temperature changing

1-D heat eqn is



$$\frac{\partial u}{x} = a^2 \frac{\partial^2 u}{\partial x^2}$$

<u>B.C</u>

(1) $u(0,t) = 50$

(2) $u(l,t) = 10$

(3) $u(x,0) = 2x + 20$

Then,

$$u(x,t) = \left(\frac{B-A}{l}\right)x + A + \left(C\cos px + D\sin px\right)e^{-\alpha^2 p^2 t}$$

$$u(x,t) = \left(\frac{10-50}{10}\right)x + 50 + \left(C\cos px + D\sin px\right)e^{-\alpha^2 p^2 t}$$

$$u(x,t) = -4x + 50 + \left(C\cos px + D\sin px\right)e^{-\alpha^2 p^2 t} \rightarrow \mathrm{I}$$

Apply (1) condition in eqn I

$$u(0,t) = 50 + C \cdot e^{-\alpha^2 p^2 t}$$

$$50 = 50 + C \cdot e^{-\alpha^2 p^2 t}$$

$$50 - 50 = C e^{-\alpha^2 p^2 t}$$

$\boxed{C = 0}$ Put $C = 0$ in eqn I

$$u(x,t) = -4x + 50 + (D\sin px)\, e^{-\alpha^2 p^2 t} \rightarrow \mathrm{II}$$

Apply (2) condition in eqn II

$$u(l,t) = -4l + 50 + (D\sin pl)\, e^{-\alpha^2 p^2 t}$$

$$10 = -4(10) + 50 + (D\sin pl)\, e^{-\alpha^2 p^2 t}$$

$$10 - 10 = D\sin pl\, e^{-\alpha^2 p^2 t}$$

$$0 = D\sin pl\, e^{-\alpha^2 p^2 t}.$$

$D \neq 0$ $\sin pl = 0$ $e^{-\alpha^2 p^2 t} \neq 0$

$$\sin pl = \sin n\pi$$

$$pl = n\pi$$

$$\boxed{p = n\pi / l}$$

Sub p in eqn II

$$u(x,t) = -4x + 50 + \left(D\sin \frac{n\pi x}{l}\right) e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t}$$

The most general solution is,

$$u(x,t) = -4x + 50 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\frac{\alpha^2 n^2\pi^2}{l^2}t} \quad \rightarrow \text{IV}$$

Apply ③ in eqn III

$t = 0$

$$u(x,0) = -4x + 50 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-0}$$

$$2x + 20 = -4x + 50 + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$6x - 30 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = f(x)$$

To find b_n:

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \int_0^l (6x - 30) \sin \frac{n\pi x}{l} \, dx$$

$u = 6x - 30$ $\quad v = \sin n\pi x/l$

$u' = 6$ $\quad v_1 = \dfrac{-\cos n\pi x/l}{n\pi/l}$

$u'' = 0$ $\quad v_2 = \dfrac{-\sin n\pi x/l}{(n\pi/l)^2}$

$$b_n = \frac{2}{l} \left[- \frac{(6x-30) \cos \frac{n\pi x}{l}}{n\pi/l} + \frac{6 \sin \frac{n\pi x}{l}}{(n\pi/l)^2} \right]_0^l$$

$$= \frac{2}{l}\left[-(6l-30)\frac{\cos n\pi}{\frac{n\pi}{l}} - 30\,\frac{\cos 0}{\frac{n\pi}{2}}\right]$$

$$= \frac{2}{l \times \frac{n\pi}{l}}\left[-(30)(-1)^n - 30\right]$$

$$= \frac{-60}{n\pi}\left[(-1)^n + 1\right]$$

$$= \begin{cases} \frac{-120}{n\pi} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

$$u(x,t) = -4x + 50 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2 \pi^2}{l^2} t}$$

2-D Heat Equation:

(i) A square plate is bounded by the lines $x=0$, $y=0$, $x=20$ and $y=20$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,20) = x(20-x)$, $0 < x < 20$ while the other three edges are kept at $0°c$. Find the steady-state temperature distribution in the plate?

Step1:

The 2D-heat flow equation in steady state is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Its square plate with lines $x=0, y=0,$ and $x=20$ and $y=20$

i) $u(0,y)=0$

ii) $u(20,y)=0$

iii) $u(x,0)=0$

iv) $u(x,20)=x(20-x)$

Then the suitable solution is

$$u(x,y)=(A\cos\lambda x+B\sin\lambda x)(ce^{\lambda y}+De^{-\lambda y}) \to \text{I}$$

i) $u(0,y)=0$

$$u(0,y)=(A\cos 0+B\sin 0)(ce^{\lambda y}+De^{-\lambda y})$$

$$0=A(ce^{\lambda y}+De^{-\lambda y})$$

$$A=0 \text{ and } (ce^{\lambda y}+De^{-\lambda y})\neq 0 \to (1)$$

Sub (1) in I we get

$$u(x,y)=B\sin\lambda x(ce^{\lambda y}+De^{-\lambda y}) \to \text{II}$$

ii) $u(20,y)=0$

$$u(20,y)=B\sin 20\lambda(ce^{\lambda y}+De^{-\lambda y})$$

$$0=B\sin 20\lambda(ce^{\lambda y}+De^{-\lambda y})$$

$$\therefore B\neq 0;\ \sin\lambda 20=0;\ ce^{\lambda y}+De^{-\lambda y}\neq 0$$

$$\sin\lambda 20=\sin n\pi$$

$$\boxed{\lambda=\frac{n\pi}{20}} \to (2)$$

Sub (2) in II we get

$$u(x,y)=B\sin\left(\frac{n\pi x}{20}\right)\left(ce^{\frac{n\pi y}{20}}+De^{\frac{-n\pi y}{20}}\right) \to \text{III}$$

iii) $u(x,0) = 0$

$$u(x,0) = B \sin \frac{n\pi x}{20} \left(c e^{0} + D e^{0} \right)$$

$$0 = B \sin \frac{n\pi x}{20} (c + D)$$

$B \neq 0$; $\sin \frac{n\pi x}{20} \neq 0$; $c + D = 0$

$\boxed{D = -c} \rightarrow 3$

Sub 3 in III we get

$$u(x,y) = B \sin\left(\frac{n\pi x}{20}\right)\left(c e^{\frac{n\pi y}{20}} - c e^{-\frac{n\pi y}{20}} \right)$$

$$= B c \sin\left(\frac{n\pi x}{20}\right)\left(e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}} \right)$$

$$\left[e^{x} - e^{-x} = 2 \sinh x \right]$$

$$\therefore u(x,y) = B c \sin\left(\frac{n\pi x}{20}\right) 2 \sinh\left(\frac{n\pi y}{20}\right) \rightarrow IV$$

$\boxed{B_n \Rightarrow 2Bc}$

The most General solution is,

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{20}\right) \sinh\left(\frac{n\pi y}{20}\right) \rightarrow V$$

iv) $u(x,20) = x(20 - x)$

$$u(x,20) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{20}\right) \sinh \frac{n\pi \cancel{20}}{\cancel{20}}$$

$$x(20-x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{20}\right) \sinh(n\pi)$$

$$B_n \sinh(n\pi) = \frac{2}{l} \int_0^l f(x) \cdot \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{20} \int_0^{20} x(20-x) \sin\left(\frac{n\pi x}{20}\right) \cdot dx$$

$$= \frac{1}{10}\int_0^{20} (20x - x^2)\ \sin\left(\frac{n\pi x}{20}\right) dx$$

$u = (20x - x^2)$ $\quad v = \sin\frac{n\pi x}{20}$

$+$

$u' = 20 - 2x$ $\quad v_1 = \dfrac{-\cos\frac{n\pi x}{20}}{\frac{n\pi}{20}}$

$-$

$u'' = -2$ $\quad v_2 = \dfrac{-\sin\frac{n\pi x}{20}}{(n\pi/20)^2}$

$+$

$u''' = 0$ $\quad v_3 = \dfrac{\cos\frac{n\pi x}{20}}{(n\pi/20)^3}$

$$= \frac{1}{10}\left(-(20x - x^2)\frac{\cos\left(\frac{n\pi x}{20}\right)}{n\pi/20} + (20 - 2x)\frac{\sin\left(\frac{n\pi x}{20}\right)}{(n\pi/20)^2} - 2\frac{\cos\frac{n\pi x}{20}}{(n\pi/20)^3}\right)_0^{20}$$

$$= \frac{1}{10}\left[\left(0 + 0 - 2(-1)^n\frac{8000}{n^3\pi^3}\right) - \left(0 + 0 - 2(1)\frac{8000}{n^3\pi^3}\right)\right]$$

$$= \frac{1}{\cancel{10}}\ 2\times\frac{800\cancel{0}}{n^3\pi^3}\left[-(-1)^n + 1\right]$$

$$B_n \sin h\, n\pi = \begin{cases} \dfrac{3200}{n^3\pi^3}\left[-(-1)^n + 1\right], & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$\therefore B_n = \begin{cases} \dfrac{3200}{n^3\pi^3 \sinh n\pi}, & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

Then the General solution is

$$u(x,y) = \sum_{n=1,3,5,\dots} \frac{3200}{n^3\pi^3 \sinh n\pi} \sin\left(\frac{n\pi x}{20}\right) \sinh\left(\frac{n\pi y}{20}\right)$$

---

2) Write the Possible solution of 2D heat eqn?

1) $u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$

2) $u(x,y) = (A\cos py + B\sin py)(Ce^{px} + De^{-px})$

3) $u(x,y) = (Ax+B)(Cy+D)$

Formula:

① Suitable soln for the steady state 2D heat equation which is periodic in x

$$u(x,y) = (A\cos px + D\sin px)(Ce^{py} + De^{-py})$$

② Suitable soln for the steady state 2D heat equation which is Periodic in y.

$$u(x,y) = (A\cos py + B\sin py)(Ce^{px} + De^{-px})$$

Then 2D heat eqn is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

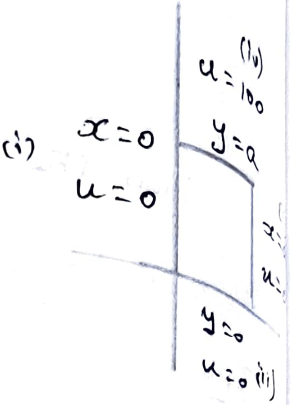
---

② A square plate is bounded by the lines $x=0$, $x=a$, $y=0$ and $y=a$. Its surfaces are insulated and the temperature along $y=a$ is kept at $100°c$. While the temperature along other three edges are at $0°c$. Find steady state temperature at any point in the plate?

The Steady - State temperature eqn :

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Then the boundary Condition are



i) $u(0,y) = 0$

ii) $u(a,y) = 0$

iii) $u(x,0) = 0$

iv) $u(x,a) = 100$

Then the Suitable soln is,

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \rightarrow I$$

i) $u(0,y) = 0$

$$u(0,y) = A(Ce^{py} + De^{-py})$$

$$0 = A(Ce^{py} + De^{-py})$$

$$A = 0 \;;\; (Ce^{py} + De^{-py}) \neq 0 \rightarrow (1)$$

put eqn (1) in I we get

$$u(x,y) = B\sin px\,(Ce^{py} + De^{-py}) \rightarrow II$$

ii) $u(a,y) = 0$

$$u(a,y) = B\sin pa\,(Ce^{py} + De^{-py})$$

$$0 = B\sin pa\,(Ce^{py} + De^{-py})$$

$$B \neq 0 \;;\; \sin pa = 0 \;;\; (Ce^{py} + De^{-py}) \neq 0$$

$$\sin pa = \sin n\pi$$

$$\boxed{p = n\pi / a} \rightarrow (2)$$

Put ② in eqn II we get

$$u(x,y) = B\sin\left(\frac{n\pi x}{a}\right)\left(Ce^{\frac{n\pi y}{a}} + De^{-\frac{n\pi y}{a}}\right) \rightarrow \text{III}$$

(iii) $u(x,0)=0 \Rightarrow$

$$u(x,0) = B\sin\left(\frac{n\pi x}{a}\right)\left(Ce^{0} + De^{0}\right) \rightarrow$$

$$0 = B\sin\left(\frac{n\pi x}{a}\right)(C+D)$$

$B \neq 0$ $\quad \sin\left(\frac{n\pi x}{a}\right) \neq 0$ $\quad (C+D)=0 \Rightarrow \boxed{D=-C} \rightarrow ③$

Put ③ in eqn III

$$u(x,y) = B\sin\left(\frac{n\pi x}{a}\right)\left(Ce^{\frac{n\pi y}{a}} - Ce^{-\frac{n\pi y}{a}}\right)$$

$$u(x,y) = BC\sin\left(\frac{n\pi x}{a}\right)\left(e^{\frac{n\pi y}{a}} - e^{-\frac{n\pi y}{a}}\right)$$

W.K.T $\left(e^{x} - e^{-x} = 2\sinh x\right)$

$$u(x,y) = BC\sin\left(\frac{n\pi x}{a}\right)\cdot 2\sinh\left(\frac{n\pi y}{a}\right)$$

The most General solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right) \rightarrow \text{IV}$$

(iv) $u(x,a)=100$

$$u(x,a) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh(n\pi)$$

$$100 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)\sinh(n\pi)$$

$$B_n \sinh n\pi = \frac{2}{l}\int_0^l f(x)\cdot\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{a}\int_0^a 100\cdot\sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{200}{a}\left[-\cos\frac{n\pi x}{a}\cdot\frac{a}{n\pi}\right]_0^a$$

$$= \frac{200\,\alpha}{\cancel{\alpha}\, n\pi}\left[-(-1)^n+1\right]$$

$$= \frac{200}{n\pi}\left[-(-1)^n+1\right]$$

$$b_n \sinh n\pi = \begin{cases} \dfrac{400}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$\therefore$ The most General solution is

$$u(x,y) = \sum_{n=1,3,5,\cdots}^{\infty} \frac{400}{n\pi \sinh n\pi} \cdot \sin\left(\frac{n\pi x}{a}\right) \cdot \sinh\left(\frac{n\pi y}{a}\right)$$

16 Mark

rod →

One Dimensional heat equation

* One Dimensional heat eqn is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

$u(x,t)$

[illegible] used in sum

* The steady steady state temp is $u = ax + b$

* The steady state temp on the rod is $u = \left(\frac{B-A}{l}\right)x + A$

* The General soln of 1D heat equation is

$$u(x,t) = \left(\frac{B-A}{l}\right)x + A + \left(C\cos px + D\sin px\right)e^{-\alpha^2 p^2 t}$$

Problem

① One end of the rod on length 10cm is kept at 30°c and other end of the rod is kept at 50°c until steady state conditions Find the steady state temperature?

A, B, $l = 10$, 30°c, 50°c

$$u = \left(\frac{B-A}{l}\right)x + A$$

$$= \left(\frac{50-30}{10}\right)x + 30$$

$$= \left(\frac{20}{10}\right)x + 30$$

$$\boxed{u = 2x + 30}$$

② A rod 30cm long has it end A and B at 20°c and 80°c until steady state condition Prevails. The temperature at each end reduced to 0°c and kept so find $u(x,t)$

20, 80, A, B, 0, $l = 30$

Before temp change

Steady state temp $u = \left(\frac{B-A}{l}\right)x + A$

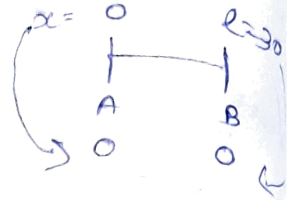
$$u = \left(\frac{80-20}{30}\right)x + 20$$

$$\boxed{u = 2x + 20}$$

After temp change

1D heat eqn is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

To find $u(x,t)$



Boundary condition

① $u(0,t) = 0$

② $u(l,t) = 0$

③ $u(x,0) = 2x + 20$

Then the correct soln is,

$u(x,t) = \left(\frac{B-A}{l}\right)x + A + (c\cos px + D\sin px)\, e^{-\alpha^2 p^2 t}$

$u(x,t) = \left(\frac{0-0}{l}\right)x + 0 + \ldots$

$u(x,t) = (c\cos px + D\sin px)\, e^{-\alpha^2 p^2 t} \rightarrow \mathrm{I}$

Apply ① condition in eqn ①

$\Rightarrow u(x,t) = (c\cos px + D\sin px)\, e^{-\alpha^2 p^2 t}$

$u(0,t) = (c\cos p(0) + D\sin p(0))\, e^{-\alpha^2 p^2 t}$

$= (c(1) + D(0))\, e^{-\alpha^2 p^2 t}$

$= c\, e^{-\alpha^2 p^2 t}$

$\Rightarrow u(0,t) = 0$ by condition ①

$c = 0$

Sub. c in eqn I

$u(x,t) = (D\sin px)\, e^{-\alpha^2 p^2 t} \rightarrow \mathrm{II}$

Apply ② condition in eqn II

$\Rightarrow u(l,t) = D\sin pl\, e^{-\alpha^2 p^2 t}$

$\Rightarrow u(l,t) = 0$ by condition ②

$\sin pl = 0 = \sin n\pi$

$\therefore pl = n\pi \Rightarrow p = \frac{n\pi}{l}$

Sub p in eqn II

$$u(x,t) = D \sin \frac{n\pi}{l} x \cdot e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \rightarrow \text{III}$$

The most general Soln is Put $D = b_n$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-\alpha^2 \frac{n^2\pi^2}{l^2} t} \rightarrow \text{III}$$

Apply 3 condition in eqn III

$$\underline{u(x,0)} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cdot e^{-\alpha^2 \frac{n^2\pi^2}{l^2}(0)} \quad (\because t=0)$$

$$= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} \cdot x \quad (\because e^0 = 1)$$

To find b_n

{Half range Sine Series formula}

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x \, dx$$

$$= \frac{2}{l} \int_0^l \overset{u}{(2x+20)} \overset{v}{\sin \frac{n\pi x}{l}} \cdot dx$$

$u = 2x + 20$ $\quad$ $v = \sin \frac{n\pi x}{l}$

$+$

$u' = 2$ $\quad$ $v_1 = \frac{-\cos n\pi x/l}{n\pi/l}$

$-$

$u'' = 0$ $\quad$ $v_2 = \frac{-\sin n\pi x/l}{n^2\pi^2/l^2}$

$$b_n = \frac{2}{l}\left[\frac{-(2x+20)\cos \frac{n\pi x}{l}}{n\pi/l} + \cancel{\frac{2\sin \frac{n\pi x}{l}}{n^2\pi^2/l^2}} \right]_0^l \quad (\sin 0 = 0)$$

$$= \frac{2}{\frac{n\pi}{\cancel{l}} \times \cancel{l}}\left[-(2l+20)\cos \frac{n\pi l}{l} + 20 \right] \quad (\cos 0 = 1)$$

$\cos n\pi = (-1)^n$

$$= \frac{2}{n\pi}\left[-(2l+20)(-1)^n + 20 \right]$$

$\because l = 30$

$$= \frac{2}{n\pi}\left[-80(-1)^n + 20\right]$$

$$\boxed{b_n = \frac{40}{n\pi}\left[-4(-1)^n + 1\right]}$$

Sub b_n in Most General soln

$$u(x,t) = \sum_{n=1}^{\infty}\left(\frac{40}{n\pi}\left[-4(-1)^n + 1\right]\right)\sin\frac{n\pi x}{l}\, e^{\frac{-\alpha^2 n^2 \pi^2}{l^2}t}$$

Unit-5

Z-transform

Z transforms are used in the Study of discrete time Signals

Defn:-

The Z-transform of the Sequence $\{f(n)\}_{n=0}^{\infty}$ is defined as $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$, if the series Converges.

If the function $f(n)$ is defined for $n = 0, 1, 2, \dots$ and $f(n) = 0$ for $n < 0$; Then $f(0), f(1), f(2), \dots$ is a sequence denoted by $\{f(n)\}$

The Sum is denoted by $F(z)$, where z is a Completed Number. This Z-transform is called one-sided z-transform.

i.e., $Z[f(n)] = F(z)$

Binomial Series Expansion:

$(1-x)^{-1} = 1 + x + x^2 + \dots$ if $|x| < 1$

$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots$ if $|x| < 1$

Logarithmic Series:

$x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = -\log_e(1-x)$ if $|x| < 1$

Exponential Series:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Z-transform of some standard sequences

1) $Z[1] = \frac{z}{z-1}, \quad |z| > 1$

W.K.T, $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

Given $f(n) = 1$

$$\therefore Z[1] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \qquad \left[(1-x)^{-1} = 1 + x + x^2 + \cdots \right.$$

$$= (1 - 1/z)^{-1} \quad \text{if } \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{(1 - 1/z)} \quad \text{if } 1 < |z|$$

$$Z[1] = \frac{z}{z-1} \quad \text{if } |z| > 1$$

Hence the proved

2) $Z[a^n] = \frac{z}{z-a}$ if $|z| > a$

W.K.T, $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

Given $f(n) = a^n$

$$\therefore Z[a^n] = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} (a \cdot z^{-1})^n$$

$$= (a \cdot z^{-1})^0 + (a \cdot z^{-1})^1 + (a \cdot z^{-1})^2 + \cdots$$

$$= 1 + az^{-1} + (az^{-1})^2 + \cdots$$

$$[(1-x)^{-1} = 1 + x + x^2 + \cdots]$$

$$= (1 - az^{-1})^{-1} \quad \text{if } |az^{-1}| < 1$$

$$= \frac{1}{(1-az^{-1})} \quad \text{if } |a| < |z|$$

$$= \frac{1}{1 - \frac{a}{z}}$$

$$Z[a^n] = \frac{z}{z-a} \quad \text{if } |z| < |a|$$

Hence The Proved

Corollary :· (Thodarchi)

If $a = 1$, $Z[1] = \frac{z}{z-1}$ if $|z| < 1$

if $a = -1$, $Z[(-1)^n] = \frac{z}{z-(-1)} = \frac{z}{z-(-1)}$

$$= \frac{z}{z+1} \quad \text{if } |z| > 1$$

③ $Z[n] = \frac{z}{(z-1)^2}, \ |z|>1, \ n \neq 1$

W.K.T, $Z[f(n)] = \sum_{n=0}^{\infty} f(n) \cdot z^{-n}$, Given $f(n)=n$

$$Z[n] = \sum_{n=0}^{\infty} n \cdot z^{-n}$$

$$= 0 + 1 \cdot z^{-1} + 2 \cdot z^{-2} + \cdots$$

$$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= \frac{1}{z} \left\{ 1 + 2 \cdot \frac{1}{z} + 3 \cdot \frac{1}{z^2} + \cdots \right\}$$

$\left[\text{W.K.T} \ (1-x)^{-2} = 1 + 2x + 3 \cdot x^2 + \cdots \right]$

$$= \frac{1}{z} \left(1 - \frac{1}{z}\right)^{-2} \quad \text{if } \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{z} \frac{1}{\left(1-\frac{1}{z}\right)^2} \quad \text{if } |1| < |z|$$

$$= \frac{1}{z} \frac{1}{\left(\frac{z-1}{z}\right)^2} \quad \text{if } |z| > 1$$

$$= \frac{z^{\cancel{2}}}{\cancel{z}(z-1)^2} \quad \text{if } |z| > 1$$

$$= \frac{z}{(z-1)^2} \quad \text{if } |z| > 1$$

Hence The Proved.

4) $Z\left[\frac{1}{n}\right] = \log_e\left(\frac{z}{z-1}\right)$, $|z|>1$, $n>0$

W.K.T, $Z[f(n)] = \sum_{n=0}^{\infty} f(n) \cdot z^{-n}$ $\left(\frac{1}{0}=\infty\right.$

$\left.\infty + \text{anything} = \infty\right)$

Given $f(n) = \frac{1}{n}$

$$Z\left[\frac{1}{n}\right] = \sum_{n=0}^{\infty} \frac{1}{n} \cdot z^{-n}$$

$$= \cancel{\frac{1}{0} z^{0}} + \frac{1}{1} \cdot z^{-1} + \frac{1}{2} \cdot z^{-2} + \dots$$

$$= \infty + z^{-1} + \frac{1}{2} \cdot z^{-2} + \frac{1}{3} \cdot z^{-3} + \dots$$

$$= z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \dots$$

$\left[\text{W.K.T } x + \frac{x^2}{2} + \frac{x^3}{3} + \dots = -\log_e(1-x)\right]$

$$= -\log_e(1 - z^{-1})$$

$$= -\log_e\left(1 - \frac{1}{z}\right)$$

$$= -\log_e\left(\frac{z-1}{z}\right)$$

$\left[x \log A = \log A^x\right.$

$-\log A = \log A^{-1}$

$\left. = \log \frac{1}{A}\right]$

$$= \log_e \frac{z}{z-1}$$

$$Z\left[\frac{1}{n}\right] = \log_e\left(\frac{z}{z-1}\right)$$

Hence the Proved

5) $Z\left[\frac{1}{n!}\right] = e^{1/z}$

W.K.T, $Z[f(n)] = \sum_{n=0}^{\infty} f(n) \cdot z^{-n}$

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

Here $[0! = 1]$

$$= \frac{1}{0!} z^{-0} + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \cdots$$

$$= 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \cdots$$

[W.K.T $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$]

$$= e^{z^{-1}}$$

$$= e^{1/z}$$

$$\therefore Z\left[\frac{1}{n!}\right] = e^{1/z}$$

Hence the proved.

⑥ $Z\left[\frac{1}{(n+1)!}\right] = z(e^{1/z} - 1)$

W.K.T $Z[f(n)] = \sum_{n=0}^{\infty} f(n) \cdot z^{-n}$

Given $f(n) = \frac{1}{(n+1)!}$

$$Z\left[\frac{1}{(n+1)!}\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n}$$

$$= \frac{1}{(0+1)!} z^{-0} + \frac{1}{(1+1)!} z^{-1} + \frac{1}{(2+1)!} z^{-2} + \cdots$$

$$= \frac{1}{1!} + \frac{1}{2!} z^{-1} + \frac{1}{3!} z^{-2} + \cdots$$

$$\left[1+\frac{x^1}{1!}+\frac{x^2}{2!}+\dots=e^x\right.$$

(or)

$$\left.\frac{x^1}{1!}+\frac{x^2}{2!}+\dots=e^x-1\right]$$

$$= z\cdot\frac{1}{z}\left[\frac{1}{1!}+\frac{1}{2!}z^{-1}+\frac{1}{3!}z^{-2}+\dots\right]$$

$$= z\cdot\left[\frac{z^{-1}}{1!}+\frac{z^{-2}}{2!}+\frac{z^{-3}}{3!}+\dots\right]$$

$$= z\cdot(e^{z^{-1}}-1) = z\cdot(e^{1/z}-1)$$

$$\therefore z\left[\frac{1}{(n+1)!}\right] = z(e^{1/z}-1)$$

---

7) $z\left[\frac{a^n}{n!}\right] = e^{a/z}$

w.k.T $z[f(n)] = \sum_{n=0}^{\infty} f(n)\cdot z^{-n}$

Given $f(n) = \frac{a^n}{(n!)}$

$$z\left[\frac{a^n}{n!}\right] = \sum_{n=0}^{\infty}\frac{a^n}{n!}\cdot z^{-n}$$

$$= \sum_{n=0}^{\infty}\frac{(az^{-1})^n}{n!}$$

$$= 1+\frac{az^{-1}}{1!}+\frac{(az^{-1})^2}{2!}+\dots$$

$$= e^{az^{-1}}$$

$$= e^{a/z}$$

$$\therefore z\left[\frac{a^n}{n!}\right] = e^{a/z}$$

Z-transform

Unilateral	Bilateral
(or)	(or)
One sided	Two Sided
(or)	(or)
one directional	Bi-directional

$X(z) = \sum_{n=0}^{\infty} f(n) z^{-n}$ or $z(f(n))$

$z(f(n)) = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$

Partial fraction Method:

1) Find the inverse z-transform of

$$\frac{z(z+1)}{(z-1)^3}$$

Soln:-

Given $F(z) = \dfrac{z(z+1)}{(z-1)^3}$ where $F(z) = z[f(n)]$

$$\therefore \frac{F(z)}{z} = \frac{z+1}{(z-1)^3} = \frac{z-1+2}{(z-1)^3}$$

$$= \frac{1}{(z-1)^2} + \frac{2}{(z-1)^3}$$

$$\therefore F(z) = \frac{z}{(z-1)^2} + 2 \cdot \frac{z}{(z-1)^3}$$

$$\Rightarrow f(n) = z^{-1}\left[\frac{z}{(z-1)^2}\right] + z^{-1}\left[\frac{2z}{(z-1)^3}\right]$$

$$= n + n(n-1)$$

$$= n^2, \quad n = 0, 1, 2, \ldots$$

Inverse by Convolution Theorem:

If $Z[f(n)] = f(z)$ and $Z[g(n)] = G(z)$,

Then $Z^{-1}[f(z) \cdot G(z)] = Z^{-1}[f(z)] * Z^{-1}[G(z)]$

$$= f(n) * g(n)$$

$$= \sum_{m=0}^{n} f(m) \cdot g(n-m)$$

1) using Convolution Theorem, find the inverse of z-transform of $\dfrac{z^2}{(z-1)(z-3)}$

Soln:-

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1} \cdot \frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= 1^n * 3^n = 3^n * 1^n$$

$$= \sum_{m=0}^{n} 3^m \cdot 1^{n-m}$$

$$= \sum_{n=0}^{n} 3^m$$

$$= 1+3+3^2+\cdots+3^n$$

$$= \frac{1-3^{n+1}}{1-3}$$

$$= \frac{3^{n+1}-1}{2}$$

2) Find $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$ using Convolution?

$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right] = Z^{-1}\left[\frac{8z^2}{2(z-1/2)4(z+1/4)}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1/2}\right] * Z^{-1}\left[\frac{z}{z-(-1/4)}\right]$$

$$= \left(1/2\right)^n * \left(-1/4\right)^n$$

$$= \left(-1/4\right)^n * \left(1/2\right)^n$$

$$= \sum_{m=0}^{n} \left(-1/4\right)^m \left(1/2\right)^{n-m}$$

$$= \sum_{m=0}^{n} \left(-1/4\right)^m \left(1/2\right)^n \left(1/2\right)^{-m}$$

$$= \left(1/2\right)^n \sum_{m=0}^{n} \left(-1/4\right)^m \cdot 2^m$$

$$= \left(1/2\right)^n \sum_{m=0}^{n} \left(-1/2\right)^m$$

$$= \left(\frac{1}{2}\right)^n \left[1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 + \dots + \left(-\frac{1}{2}\right)^n\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1-\left(-\frac{1}{2}\right)^{n+1}}{1-\left(-\frac{1}{2}\right)}\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1-\left(-\frac{1}{2}\right)^{n+1}}{\frac{3}{2}}\right]$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n \left[1-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^n\right]$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n \left[1+\frac{1}{2}\left(-\frac{1}{2}\right)^n\right]$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(-\frac{1}{4}\right)^n \quad n = 0, 1, 2, \dots$$

Difference equations:

1) Form a difference equation by eliminating arbitrary constant $u_n = A\,2^{n+1}$

Given $u_n = A\,2^{n+1} \rightarrow (1)$

$u_{n+1} = A\,2^{n+2} \rightarrow (2)$

$$\frac{(2)}{(1)} \Rightarrow \frac{u_{n+1}}{u_n} = \frac{A\,2^{n+2}}{A\,2^{n+1}}$$

$$= 2$$

$$\therefore u_{n+1} = 2u_n$$

This is a 1^{st} order difference Equation

2) Find the equation generated by

$y_n = an + b\,2^n$

Given $y_n = an + b\,2^n \rightarrow (1)$

$y_{n+1} = a(n+1) + b\,2^{n+1} \rightarrow (2)$

and $y_{n+2} = a(n+2) + b\,2^{n+2} \rightarrow (3)$

Eliminating a and b, using (1), (2), (3)

we get

$$\begin{vmatrix} n & 2^n & y_n \\ n+1 & 2^{n+1} & y_{n+1} \\ n+2 & 2^{n+2} & y_{n+2} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} y_n & n & 2^n \\ y_{n+1} & n+1 & 2^{n+1} \\ y_{n+2} & n+2 & 2^{n+2} \end{vmatrix} = 0$$

Taking out 2^n From C_3

$$\Rightarrow 2^n \begin{vmatrix} y_n & n & 1 \\ y_{n+1} & n+1 & 2 \\ y_{n+2} & n+2 & 2^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} y_n & n & 1 \\ y_{n+1} & n+1 & 2 \\ y_{n+2} & n+2 & 4 \end{vmatrix}$$

Expanding by C_1, we get

$$y_n[4(n+1)-2(n+2)] - y_{n+1}[4n-(n+2)] +$$

$$y_{n+2}[2n-(n+1)] = 0$$

$$\Rightarrow y_n[2n] - y_{n+1}(3n-2) + y_{n+2}(n-1) = 0$$

$$\Rightarrow (n-1)\, y_{n+2} - (3n-2)\, y_{n+1} + 2n y_n = 0$$

There is a difference equation of order 2

Application of z-transform to solve linear difference equations:

The General form of a linear difference eqn of r^{th} order in the sequence y_n is.

$$a_0 y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + \dots + a_r y_n = f(n) \rightarrow (1)$$

when $n = 0, 1, 2, \dots$ and $a_0, a_1, \dots, a_r$ are Constants

To solve the difference eqn apply Z-transform on both sides using the formulae

$$Z[y_{n+1}] = z[F(z) - y_0] \quad \text{where } Z[y_n] = F(z)$$

$$Z[y_{n+2}] = z^2\left[F(z) - y_0 - \frac{y_1}{z}\right]$$

$$Z[y_{n+3}] = z^3\left[F(z) - y_0 + \frac{y_1}{z} - \frac{y_2}{z^2}\right]$$

Problem's based on solution of difference equation:

1) Solve $y_{n+1} - 2y_n = 0$ gives $y_0 = 3$

Given $y_{n+1} - 2y_n = 0$

Applying Z-transforms on both sides, we get

$Z[y_{n+1}] - 2Z[y_n] = 0$

Let $Z[y_n] = F(z)$

$z[F(z) - y_0] - 2F(z) = 0$

$(z-2)\,F(z) - 3z = 0$

$\Rightarrow F(z) = \dfrac{3z}{z-2}$

$\Rightarrow Z[y_n] = \dfrac{3z}{z-2}$

Taking inverse Z-transform,

$y_n = 3Z^{-1}\left[\dfrac{z}{z-2}\right] = 3\cdot 2^n, \; n = 0, 1, 2, \ldots$

$\left[\because Z^{-1}\left(\dfrac{z}{z-a}\right) = a^n\right]$

2) Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$ using Z-transform.

Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

Applying Z-transform on both sides, we get

$Z[y_{n+2}] + 6Z[y_{n+1}] + 9Z[y_n] = Z[2^n]$

Let $z[y_n] = F(z)$

$$\therefore z^2\left[F(z) - y_0 - \frac{y_1}{z}\right] + 6z\left[F(z) - y_0\right] + 9F(z) = \frac{z}{z-2}$$

Given $y_0 = y_1 = 0$

$$\Rightarrow z^2[F(z) - 0 - 0] + 6z[F(z) - 0] + 9F(z) = \frac{z}{z-2}$$

$$\Rightarrow (z^2 + 6z + 9)F(z) = \frac{z}{z-2}$$

$$\Rightarrow F(z) = \frac{z}{(z-2)(z^2+6z+9)}$$

Let $$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$$

$$\therefore 1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

Put $z = -3$, $1 = C(-3-2) \Rightarrow C = -1/5$

Put $z = 2$, $1 = A(2+3)^2$

$$\Rightarrow A = 1/25$$

Equating Coefficients of z^2

$$0 = A + B \Rightarrow B = -A = -1/25$$

$$\therefore F(z) = \frac{1}{25(z-2)} - \frac{1}{25(z+3)} - \frac{1}{5(z+3)^2}$$

$$F(z) = \frac{1}{25}\frac{z}{(z-2)} - \frac{1}{25}\frac{z}{(z+3)} - \frac{1}{5}\frac{z}{(z+3)^2}$$

$$\Rightarrow z[y_n] = \frac{1}{25}\frac{z}{(z-2)} - \frac{1}{25}\frac{z}{(z+3)} - \frac{1}{5}\frac{z}{(z+3)^2}$$

Taking inverse z-transform

$$y_n = \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5} z^{-1}\left[\frac{z}{(z+3)^2}\right]$$

$$= \frac{1}{25}\cdot 2^n - \frac{1}{25}(-3)^n - 1/5\left(-1/3\right) z^{-1}\left[\frac{-3z}{z-(-3)^2}\right]$$

$$y_n = 1/25 \cdot 2^n - \frac{1}{25}(-3)^n + \frac{1}{5} n(-3)^n, \quad n = 0,1,2,\ldots$$

Difference Equation:

i) $z[y(n+3)] = z^3 F(z) - z^3 y(0) - z^2 y(1) - z y(2)$

ii) $z[y(n+2)] = z^2 F(z) - z^2 y(0) - z y(1)$

iii) $z[y(n+1)] = z F(z) - z y(0)$

iv) $z[y(n)] = F(z)$

1) Solve $y(n+2) + 3y(n+1) + 2y(n) = 0$ given that $y(0) = 1$, $y(1) = 2$

Soln:

$$y(n+2) + 3y(n+1) + 2y(n) = 0$$

$$z[y(n+2)] + 3z[y(n+1)] + 2z[y(n)] = 0$$

$$[z^2 F(z) - z^2 y(0) - z y(1)] + 3[zF(z) - zy(0)] + 2F(z)$$

$$[z^2 F(z) - z^2(1) - z(2)] + 3[zF(z) - z(1)] + 2F(z) = 0$$

$$z^2 F(z) - z^2 - 2z + 3zF(z) - 3z(1) + 2F(z) = 0$$

$$z^2 F(z) + 3zF(z) + 2F(z) = z^2 + 2z + 3z$$

$$F(z)[z^2 + 3z + 2] = z^2 + 5z$$

$\frac{2}{+2}$ / 3

$$F(z)\left[(z+1)(z+2)\right] = z^2+5z$$

$$F(z) = \frac{z^2+5z}{(z+1)(z+2)} = \frac{z(z+5)}{(z+1)(z+2)}$$

$$\therefore \frac{F(z)}{z} = \frac{(z+5)}{(z+1)(z+2)} \rightarrow (1)$$

Now

$$\frac{z+5}{(z+1)(z+2)} = \frac{A}{(z+1)} + \frac{B}{(z+2)} \rightarrow (2)$$

$$\frac{z+5}{(z+1)(z+2)} = \frac{A(z+2)+B(z+1)}{(z+1)(z+2)}$$

$$z+5 = A(z+2)+B(z+1) \rightarrow (3)$$

Put $z=-1$ in eqn (3)

$$-1+5 = A(-1+2)+B(-1+1)$$

$$4 = A(1)+B(0)$$

$$\boxed{A=4}$$

Put $z=-2$ in eqn (3)

$$-2+5 = A(-2+2)+B(-2+1)$$

$$3 = A(0)+B(-1)$$

$$3 = -B.$$

$$\therefore \boxed{B=-3}$$

Now eqn (2), $\dfrac{z+5}{(z+1)(z+2)} = \dfrac{4}{(z+1)} + \dfrac{(-3)}{(z+2)}$

$$\frac{F(z)}{z} = \frac{4}{(z+1)} - \frac{3}{(z+2)}$$

$$F(z) = 4\left[\frac{z}{z+1}\right] - 3\left[\frac{z}{z+2}\right]$$

Taking z^{-1} on both sides

$$z^{-1}[F(z)] = 4z^{-1}\left[\frac{z}{z+1}\right] - 3z^{-1}\left[\frac{z}{z+2}\right]$$

$$y(n) = 4(-1)^n - 3(-2)^n$$

(2) Solve $y(n+3) - 3y(n+1) + 2y_n = 0$ given

That $y(0) = 4$, $y(1) = 0$, $y(2) = 8$

Soln:

$$y(n+3) - 3y(n+1) + 2y_n = 0$$

$$Z[y(n+3)] - 3Z[y(n+1)] + 2Z[y(n)] = 0$$

$$[z^3 F(z) - z^3 y(0) - z^2 y(1) - z\,y(2)] - 3[zF(z) - zy(0)]$$

$$+ 2F(z) = 0$$

$$[z^3 F(z) - z^3 (4) - z^2(0) - z(8)] - 3[zF(z) - z(4)]$$

$$+ 2F(z) = 0$$

$$[z^3 F(z) - z^3 4 - z8] - 3zF(z) + 3z(4) + 2F(z) = 0$$

$$z^3 F(z) - 3zF(z) + 2F(z) = 4z^3 + 8z - 12z$$

$$z^3 F(z) - 3zF(z) + 2F(z) = 4z^3 - 4z$$

$$F(z)[z^3 - 3z + 2] = 4z^3 - 4z$$

$$z_1 = -2;\ z_2 = 1,\ z_3 = 1$$

$$F(z)[(z+2)(z-1)(z-1)] = 4z^3 - 4z$$

$$F(z) = \frac{4z(z^2-1^2)}{(z+2)(z-1)^2}$$

$$\frac{F(z)}{z} = \frac{4(z^2-1^2)}{(z+2)(z-1)^2}$$

$(a^2-b^2) = (a+b)(a-b)$

$(z^2-1^2) = (z+1)(z-1)$

$$\frac{F(z)}{z} = \frac{4(z+1)\cancel{(z-1)}}{(z+2)(z-1)^{\cancel{2}}} = \frac{4(z+1)}{(z+2)(z-1)} \rightarrow (1)$$

$$\frac{4(z+1)}{(z+2)(z-1)} = \frac{A}{(z+2)} + \frac{B}{(z-1)} \rightarrow (2)$$

$$\frac{4(z+1)}{(z+2)(z-1)} = \frac{A(z-1)+B(z+2)}{(z+2)(z-1)}$$

$$4(z+1) = A(z-1) + B(z+2) \rightarrow (3)$$

Put $\boxed{z=-2}$ in eqn (3)

$$4(-2+1) = A(-2-1) + B(-2+2)$$

$$4(-1) = A(-3) + B(0)$$

$$\frac{-4}{-3} = A \Rightarrow \boxed{A = \frac{4}{3}}$$

Put $\boxed{z=1}$ in eqn (3)

$$4(1+1) = A(1-1) + B(1+2)$$

$$8 = B(3)$$

$$\boxed{B = \frac{8}{3}}$$

Now eqn 2 becomes,

$$\frac{4(z+1)}{(z+2)(z-1)} = \frac{\frac{4}{3}}{(z+2)} + \frac{\frac{8}{3}}{(z-1)}$$

$$\frac{F(z)}{z} = \frac{\frac{4}{3}}{(z+2)} + \frac{\frac{8}{3}}{(z-1)}$$

$$F(z) = \frac{4}{3}\left(\frac{z}{z+2}\right) + \frac{8}{3}\left(\frac{z}{z-1}\right)$$

$$z^{-1}(F(z)) = \frac{4}{3} z^{-1}\left(\frac{z}{z+2}\right) + \frac{8}{3} z^{-1}\left(\frac{z}{z-1}\right)$$

$$y(n) = \frac{4}{3}(-2^n) + \frac{8}{3}(1^n)$$